INSTABILITIES IN VISCOELASTIC SOLIDS UNDERGOING FINITE DEFORMATION

by

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TO MY PARENTS

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CERTIFICATE

This is to certify that the research carried out by Subrata Sengupta for the preparation of the thesis, 'Instabilities in Viscoelastic Solids Undergoing Finite Deformation', has been supervised by me. The thesis is being submitted to the Department of Civil Engineering, Indian Institute of Technology Kanpur, in partial fulfilment of the requirements for the degree of Doctor of Philosophy, and has not been submitted for a degree elsewhere.

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Subrata Sengupta

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NOMENCLATURE

All symbols are defined whenever they appear first in the text.

A, B, C	-	Expressions defined in (5.15, 5.15a, 5.15b, 5.15c) pages 92,93, Ch.5.
A_1, A_2, A_3, A_4	-	Constants defined by eq. (3.19), page 59, Ch.3
A_1, A_2, A_3	-	Constants defined by eq. (4.12), page 73, Ch.4
B	-	Body in current configuration
B^o	-	Body in reference (initial) configuration
E	-	Modulus (Elastic) as defined in page 13, Ch.1
E'	-	Expression defined in page 94, Ch.5
F	-	Modulus (Viscous) as defined in page 12, Ch.1
F_{ijkl}	-	Defined by eq. (1.33), page 16, Ch.1
I_o, I_1	-	Modified Bessel Functions of first kind, order zero and one respectively, page 60 , Ch.3
J_o, J_1	-	Bessel Functions of first kind, order zero and one respectively, page 59, Ch.3
K_{ijkl}	-	Instanteneous viscoelastic moduli
L	-	Operator as defined in page 57, Ch. 3
$2L$	-	Current length of the cylinder, Ch.3, Ch.4
$2L^o$	-	Initial length of the cylinder, Ch.3, Ch.4
P_J	-	Load vector in the reference (initial) configuration
P^o_J	-	Load vector in current configuration
Q	-	Instantaneous material parameter (elastic) defined by eq. (1.35), page 16, Ch.1

Q_{ijkl}	-	Instanteneous elastic moduli as defined in (1.32), page 16, Ch.1
R	-	Current mean radius of the thin shell in Ch.4
S	-	Surface area in the final position
S^o	-	Surface area in the initial position
S'	-	Expression defined in page 92, Ch.5
S_T	-	Surface area on which tractions are prescribed
S_v	-	Surface area on which velocities are prescribed
T_j	-	True traction
T^o_j	-	Nominal traction
$\dot{T}_j$	-	Nominal traction-rate
$\bar{T}$	-	Defined by the expression page 74, Ch.4
X_i	-	Coordinate system in the reference (initial) configuration, Ch.1, in suffix notation
Y_1	-	Bessel function of second kind, order one, Ch.3
a_o, b_o	-	Initial dimensions of the rectangular specimen Ch.2
	-	Initial dimensions of the rectangular plate Ch.5
a, b	-	Current dimensions of the rectangular specimen Ch.2
	-	Current dimensions of the rectangular plate Ch.5
c_o	-	Third dimension of the rectangular specimen Ch.2
e	-	$(\alpha\bar{\omega}^2/k^2)$ / $(1 + \lambda\bar{\omega}-\theta)$ in page 58, Ch.3

f_1, f_2, f_3	-	Function of r defined eq. (3.11), page 54, Ch.3
$f(y)$	-	Function of y in Ch.2
$g(y)$	-	Function of y in Ch.2
h	-	Current thickness of the rectangular plate Ch.5
i	-	$\sqrt{-1}$
k	-	$n\pi/a$ in Ch.2
	-	$n\pi/L$ in Ch. 3
	-	$n\pi/b$ in Ch. 5
m	-	Wave number in z direction Ch. 4
	-	Wave number in x direction Ch. 5
n	-	Wave number in x direction Ch. 2
	-	Wave number in z direction Ch.3, Ch.4
	-	Wave number in y direction Ch. 5
n_i^o	-	Unit outward normal to the surface S^o
n_i	-	Unit outward normal to the surface S
s_{ij}	-	Lagrangian or nominal stress tensor
$\dot{s}_{ij}$	-	Nominal stress rate
t	-	Mean thickness of the cylinder Ch. 5
$\bar{t}$	-	$t^2/12R^2$, Ch.5
u, v, w	-	Velocity components in r, θ, z or x, y, z directions respectively
v_i	-	Velocity components in x_i directions
x_i	-	Coordinate system in suffix notation

α	-	ρ/Q
β	-	Roots of the characteristics equations, Ch.2, Ch.3
β_i	-	Roots of the characteristics equation, Ch.2, Ch.3
$\tilde{\beta}_i$	-	Roots of the characteristics equation, Ch.2
$\bar{\beta}$	-	Roots of the characteristics equation, Ch.2
γ	-	Real number $\beta = \gamma i$ in Ch.2
δ_{ij}	-	Kronecker delta
$\frac{\delta}{\delta t}$	-	Oldroyd or convected derivative
$\frac{D}{Dt}$ or $(\dot{})$	-	Material derivative
$\frac{\mathcal{D}}{\mathcal{D}t}$	-	Jaumann derivative
θ	-	Nondimensional load parameter $=\sigma/2Q$ in Ch.2, Ch.3 $=\tau/2Q$ in Ch.4
θ_1	-	Nondimensional load parameter $\sigma_1/2Q$ in x direction, Ch.5
θ_2	-	Nondimensional load parameter $\sigma_2/2Q$ in y direction, Ch. 5
σ	-	Stress when other stress components are zero, Ch.2, Ch.3
σ_1	-	Stress in the x direction in Ch.5
σ_2	-	Stress in the y direction in Ch.5
σ_a	-	Stress in spring, Ch.1
σ_b	-	Stress in the dashpot , Ch.1
σ_{ij}	-	True(Cauchy) stress tensor
τ	-	Shear stress in Ch.4
τ_{ij}	-	Kirchoff stress tensor

Δ_1, Δ_2	-	Determinants in Ch.2
η	-	Viscosity
η_{ijkl}	-	Instantaneous material (viscous) parameters as defined in eq. (1.33), Ch.1
ξ	-	$(r - R)$
Ψ	-	$\frac{n\pi R}{L}$
e_a	-	Strain in the spring, Ch.1
e_b	-	Strain in the dashpot, Ch.1
e_{ij}	-	Strain-rate tensor
ω	-	Frequency
ω_{ij}	-	Rate of rotation tensor
ν	-	$m\pi/a$ in Ch.5
ρ^o	-	Initial density
ρ	-	Current density
$\varphi(y)$	-	Function of y in Ch.2
λ	-	η/Q
λ_1	-	Nondimensional viscosity parameter
$\bar{\lambda}$	-	Material parameter defined in eq.(1.33), page 13, Ch.1
Ω	-	Non-dimensional frequency parameter
r,θ,z	-	Cylindrical coordinate system
x,y,z	-	Cartesian coordinate system
x_1,x_2,x_3	-	Coordinate system in suffix notation.

SYNOPSIS

The problem of stability in a finitely deformed solid can be studied either by using the method of superposing small peturbations on large deformations or, by employing the stability criterion with the help of variational technique. The thesis presents a detailed study of the following four cases:

(i) Instabilities in a rectangular viscoelastic solid under axial load in a state of plane strain.

(ii) Axisymmetric instabilities in a viscoelastic circular cylinder under axial loading.

(iii) Stability of a thin-walled viscoelastic cylinder under finite twist.

(iv) Stability of a thin rectangular viscoelastic plate under in-plane loading.

The chapter-wise break-up of the thesis is as follows:

Chapter one presents a brief literature survey in the subject of viscoelasticity and stability in a visco-elastic continuum. It includes a complete formulation of the rate boundary value problem and presents the derivation

of the stability criterion based on the Kelvin-Dritchlet concept of stability of a system in motion. Starting from a general viscoelastic material, the relevent constitutive relations are obtained in the rate form and are finally specialized for the case of an incompressible viscoelastic material which this thesis is chiefly concerned with.

In chapter two, the stability phenomenon of a rectangular viscoelastic solid under plane strain in compression/tension is analysed. By solving the governing differential equation of motion and studying the variation of the resulting displacement with time, the critical conditions are obtained for instability. A detailed indepth study of the behaviour of the resulting characteristic equations is presented which makes the discussion on the stability phenomenon more rational and complete. It is found that the instability in the symmetric mode is ruled out completely. The results for a corresponding elastic material are obtained as a special case ; these results are similar to those obtained by earlier researchers but the manner in which they are obtained is mathematically (and also logically) more rigorous.

Stability of a viscoelastic solid cylinder of finite dimensions under axial load is examined in chapter three. Using the formulation developed in chapter one and following the approach adopted by Cheng et al. (1971) and Kumar and Niyogi (1982) for an elastic material, a condition is derived for the existence of a non-trivial solution. Only the axisymmetric mode of instability is considered. The instability condition is numerically solved to obtain the critical stress for different values of cylinder dimensions.

Chapter four is devoted to the investigation of stability of a thin-walled long circular viscoelastic cylindrical shell under finite twist. The stability criterion presented in chapter one is used to derive the condition for stability. The approach is similar to that of Neale (1973) and is enlarged to include the case of viscoelastic materials. By studying the stress-frequency relationship, the critical shear stress at instability is obtained and its variation with radius-thickness ratio of the shell is shown graphically.

Using the technique, similar to that in chapter four, the stability of a thin rectangular viscoelastic plate under inplane loading is examined in chapter five. The material is considered incompressible and the plate is subjected to

either axial or biaxial loads. The method of solution is similar to that of Sewell (1973) but the dynamic effects are included in order to study the time-dependent behaviour of the material. The critical stress at instability is obtained for simply supported edge conditions. When the loading is biaxial, the interaction curves are plotted for different aspect ratios of the plate.

The discussion on the results for each investigation are placed at the end of the corresponding chapter. However, the general observations and all results are summarized in chapter six.

CHAPTER : 1

CHAPTER : 1

INTRODUCTION

1.1 PRELIMINARY REMARKS

No mathematical theory can completely describe the complex world around us. Every theory is aimed at a certain class of phenomena, formulates their essential features, and disregards what is of minor importance. The theory meets its limits of applicability where a disregarded influence becomes important. Thus, rigid-body dynamics describes in many cases the motion of actual bodies, but it fails to produce more than a few general statements in the case of impact, because elastic or inelastic deformation, no matter how local or how small, attains a dominating influence.

For a long time mechanics of deformable bodies has been based upon Hooke's law, i.e. upon the assumption of linear elasticity. It is well known that most engineering materials like metals, concrete, wood, soil, are not linearly elastic or, are so within limits too narrow to cover the range of practical interest. Nevertheless, almost all routine stress analysis is still based on Hooke's law because of its simplicity.

In the course of time engineers have become increasingly conscious of the importance of the inelastic behaviour of

many materials, and mathematical formulations have been attempted and applied to practical problems. Outstanding among them are the theories of ideally plastic and of viscoelastic materials. While plastic behaviour is essentially nonlinear (piecewise linear at best), viscoelasticity, like elasticity, permits a linear theory. The investigation of stability in linear viscoelastic material is the subject of the present thesis. Let us first briefly review this vast subject of stability.

1.2 BRIEF SUBJECT REVIEW

In investigating the stability of a structure, one may ask the following questions:

i) If the system, which is initially in an equilibrium state, is given an arbitrary small disturbance, does it remain near the equilibrium state? If so, the system is stable.

ii) Does it remain near the equilibrium state, and in addition tend to return to the equilibrium state? If it does, the system is asymptotically stable.

iii) What bounds must be placed on the magnitude of the initial disturbances so that, given a disturbance within these bounds, the system will eventually regain its original equilibrium state? In other words what is the extent of asymptotic stability?

In practical sense, an equilibrium configuration of a mechanical system is said to be stable if accidental forces, shocks, vibrations, eccentricities, imperfections, inhomogenieties, residual stresses or other probable irregularities do not cause the system to depart excessively or disastrously from that configuration. In a mathematical sense, stability is usually interpreted to mean that infinitesimal disturbances will cause only infinitesimal departure from the given equilibrium configuration. The mere fact that an assemblage is stable in this refined sense, does not necessarily signify that it is safe from engineering viewpoint.

There are two methods of stability analysis, one is the static method and the other is the dynamic one. An equilibrium state is considered stable by the static criterion if no adjacent equilibrium (configuration) exists, or if the overall potential energy (of the system) is a relative minimum. The dynamic method is more general in the sense that all the approaches, based on the static concept, are the special cases of this approach when inertia forces are neglected. Since this method takes into account the inertia forces in its formulation, the mass distribution of the system becomes as important as the elastic stiffness of the system. The response of the system, therefore, becomes a function of both

the space and the time coordinate. The method consists of the following stipulations and concepts: an unperturbed state whose stability is being studied is specified. A perturbation is then applied to the unperturbed state so that it is transformed into a perturbed one. Certain characteristics, called the norms, are emphasized which define the states at any desired time. The change in the norms during the transition from the unperturbed state to the perturbed one under the influence of perturbation is determined. Based on this behaviour, a conclusion may be reached regarding the stability of the unperturbed state, or its instability.

In examining the stability of structures undergoing large deformations, usually two methods have been adopted. The first is the method of small perturbations superimposed on finite deformation as available in, for example, Biot (1965). A system of governing differential equations of motion is obtained; these equations together with rate boundary conditions and rate material properties constitute the rate problem. The second method makes use of a variational technique in which a certain functional vanishes in the absence of a stable solution ; non-vanishing of the functional implies stability. A detailed account of both the approaches, especially for quasi-static problems in continuum mechanics, is contained in Hill's work (1959, 1961).

The incremental method of analysis developed by Biot in the context of the theory of elasticity could be extended to analyse the stability problems of viscous and viscoelastic media. In fact, this realization opened an entirely new phase in the problems of deformation of the earth's crust and tectonic folding of geological structures (Biot, 1965). The mathematical foundations of the constitutive equations of linear and nonlinear viscoelastic materials have been derived by many authors. It would be sufficient to mention only a certain group of papers on this subject, namely those by Rivlin and Erickson (1955), Rivlin (1955), Spencer and Rivlin (1959) , Noll (1955, 1958), Coleman and Noll (1960). The general theory of nonlinear viscoelastic materials with memory has been formulated by Green and Rivlin (1957, 1960) and by Green et al. (1959). Some aspects of small deformations superimposed on large deformations in materials with fading memory have been discussed by Pipkin and Rivlin (1961). Zahorski (1965a) dealt with the problem of small additional motion superposed on fundamental slow deformation of nonlinear viscoelastic media and applied the theory to the problem of buckling of an incompressible column of arbitrary dimensions (1965b). Dost and Glockner (1982) also investigated the dynamic stability of a geometrically perfect three element linear viscoelastic column using an approximate numerical technique.

In fact, very little work is available on the systematic investigation of stability of some basic problems in a finitely deformed viscoelastic medium. The reason, perhaps, has been the difficulty in handling the complex mathematical formulation available. It has not been realized so far that the Hill's theory (1961) of instability and bifurcation in elastic and elastic-plastic continuum, which has been found so useful for application in variety of problems (see e.g. Sewell,1972) can be suitably modified to include even viscoelastic materials and that the problem can be made identical to one of investigating nonconservative problems in the theory of elastic stability by the conventional dynamic method. This thesis takes up this challenging work and shows the application of the theory in four basic problems (Chapters two to five), as summarised in the Synopsis.

1.3 RATE PROBLEM

In the rate problem the stress distribution, the deformation state and the constitutive material properties are supposed to be known at a generic instant and the task is to determine the ensuing motion of the body under an incremental change of the external loading.

A common type of boundary conditions for a mixed problem is that the rate of change of nominal load vector, which is independent of any structural geometry changes, is

prescribed on some part S_T of the surface of the body and particle velocities on the remaining surface S_v. As the current traction is then related to unit reference area, it is convenient to introduce the nominal (or, the Lagrangian or, the first Piola-Kirchhoff) stress tensor.

Referring to a fixed coordinate system, which for simplicity is taken as Cartesian here (Figure 1.1), the coordinates of a particle orginally at X_i in some reference (initial) configuration B^o is denoted by x_i in the current configuration B at time t. The vector areas of the same surface element are then denoted by $n_i^o dS^o$ and $n_i dS$ in the two configurations, respectively. The current load vector acting on a surface element is

$$dP_i^o = T_i^o \, dS^o \qquad \dots (1.1)$$

where T_i^o is the nominal traction.

The nominal stress tensor s_{ij} is then defined by

$$T_j^o = n_i^o \, s_{ij} \qquad \dots (1.2)$$

This means that s_{ij} denotes the current j-th component of the load per unit reference area, acting on a surface element which has the normal in the i-th direction in the reference configuration.

Now the true traction T_i acting on the surface element is given by

$$dP_i = T_i \, dS \qquad \ldots (1.3)$$

where by Cauchy's stress hypothesis

$$T_j = n_i \, \sigma_{ij} \qquad \ldots (1.4)$$

σ_{ij} being the true (or the Cauchy) stress tensor. Then

$$n_i \, \sigma_{ij} \, dS = n_k^o \, s_{kj} \, dS^o \qquad \ldots (1.5)$$

The vector areas of the surface element in the two configurations are related through the basic formula (cf. e.g. Storåkers, 1973)

$$n_l \, dS = \frac{\rho^o}{\rho} \cdot \frac{\partial X_i}{\partial x_l} \cdot n_i^o \, dS^o \qquad \ldots (1.6)$$

where ρ^o and ρ are the mass densities in the reference and the current configurations respectively.

Introduction of (1.6) into (1.5) then yields the basic relation

$$s_{ki} = \frac{\rho^o}{\rho} \; \sigma_{ij} \, \frac{\partial X_k}{\partial x_j} \qquad \ldots (1.7)$$

Again

$$dP_j = \tau_{ij}(n_i^o \, dS^o) \qquad \ldots (1.8)$$

where τ_{ij} is the Kirchhoff stress (or second Piola-Kirchhoff stress) tensor. It can be shown that

$$\sigma_{ij} = \frac{\rho}{\rho o} \tau_{ij} \qquad \ldots (1.9)$$

Similarly, one obtains

$$s_{ij} = \frac{\partial X_j}{\partial x_k} \tau_{ik} \qquad \ldots (1.10)$$

If the coordinate x_i is chosen to coincide with X_i at the current instant

$$\left.\begin{array}{l} X_i \equiv x_i \;, \qquad \rho = \rho^o \\ \text{such that} \\ \sigma_{ij} = \tau_{ij} = s_{ij} \end{array}\right\} \qquad \ldots (1.11)$$

Let us now consider the different stress-rates which are commonly used. First is the material derivative, i.e. the time derivative following the element, denoted by $\frac{D(\;)}{Dt}$ or simply by $(\dot{\;})$. Again

$$\frac{D(\;)}{Dt} = \frac{\partial(\;)}{\partial t} + v_i \frac{\partial(\;)}{\partial x_i} \qquad \ldots (1.12)$$

where v_i are the velocity components.

Other two most commonly used stress-rates are the Oldroyd or the convected derivative (Oldroyd, 1950), which is associated with the axes fixed in the body and also deforming with it, and the Jaumann derivative which is

associated with the body and rotating with the body, but not deforming with it. The former is denoted by $\frac{\delta(\ )}{\delta t}$ and the later by $\frac{\mathcal{D}(\ )}{\mathcal{D} t}$. They are expressed as

$$\frac{\delta(\ )_{ij}}{\delta t} = (\dot{\ })_{ij} - (\ )_{ik}\, v_{j,k} - (\ )_{jk}\, v_{i,k} \quad \ldots (1.13)$$

$$\frac{\mathcal{D}(\ )_{ij}}{\mathcal{D}\tau} = (\dot{\ })_{ij} - (\ )_{ik}\, \omega_{jk} - (\ )_{jk}\, \omega_{ik} \quad \ldots (1.14)$$

where $\omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ is the antisymmetric part of the velocity gradient tensor and a comma signifies partial differentiation with respect to X_i.

Equations of Motion and Boundary Conditions

Neglecting body forces the condition for balance of momentum is

$$\int_{S^o} T_j^o \, dS^o = 0 \quad \ldots (1.15)$$

in the reference (initial) configuration, or introducing s_{ij}

$$\int_{S^o} s_{ij}\, n_i^o \, dS^o = 0 \quad \ldots (1.16)$$

Application of Gauss' divergence theorem then yields the field equations for equilibrium of the current state

$$\frac{\partial s_{ij}}{\partial X_i} = 0 \quad \ldots (1.17)$$

and for continued balance

$$\frac{\partial \dot{s}_{ij}}{\partial X_i} = 0 \qquad \ldots \quad (1.18)$$

where a dot denotes material rate of change.

When the inertia forces are considered, the corresponding equation of motion in the incremental form, will be

$$\frac{\partial \dot{s}_{ij}}{\partial X_i} = \rho \frac{\partial^2 v_j}{\partial t^2} \qquad \ldots \quad (1.19)$$

where v_j are the particle velocities. In fact, v_i should be interpreted as the incremental displacement so that, at the onset, the objection in units of the r.h.s. of (1.19) does not arise.

It follows from the law of balance of moment of momentum applied to the current configuration that the true (Cauchy) stress tensor is symmetric, which implies for s_{ij} through (1.7)

$$s_{mi} \frac{\partial x_j}{\partial X_m} = s_{nj} \frac{\partial x_i}{\partial X_n} \qquad \ldots \quad (1.20)$$

Material derivation then yields

$$\dot{s}_{mi} \frac{\partial x_j}{\partial X_m} + s_{mi} \frac{\partial v_j}{\partial x_k} \cdot \frac{\partial x_k}{\partial X_m} = \dot{s}_{nj} \frac{\partial x_i}{\partial X_n} + s_{nj} \frac{\partial v_i}{\partial x_l} \cdot \frac{\partial x_l}{\partial X_n} \qquad \ldots \quad (1.21)$$

where v_i are the particle velocities. Material derivation may be performed with respect to any monotonically increasing parameters which does not have to be real time except when dynamic effects are considered.

When formulating the rate problem it is convenient for the present purpose to let the reference (initial) configuration and the current configuration coincide at the generic instant. The equations of motion (1.19) then simplify to

$$\dot{s}_{ij,i} = \rho \ddot{v}_j \qquad \ldots \quad (1.22)$$

and

$$\dot{s}_{ij} + s_{kj} v_{i,k} = \dot{s}_{ji} + s_{li} \cdot v_{j,l} \qquad \ldots \quad (1.23)$$

where a comma denotes partial differentiation.

With this choice of reference frame the true and nominal stresses coincide at the considered instant although from (1.7) their material rates of change differ through

$$\dot{s}_{ij} = \dot{\sigma}_{ij} + \sigma_{ij} v_{k,k} - \sigma_{jk} v_{i,k} \qquad \ldots \quad (1.24)$$

When the nominal surface traction rate is prescribed on S_T and the particle velocities are specified on S_v, the dynamical boundary conditions are

$$\left.\begin{aligned} \dot{T}_j &= n_i \, \dot{s}_{ij} \quad \text{on } S_T \\ v_j &= V_j \qquad \text{on } S_v \end{aligned}\right\} \qquad \ldots \quad (1.25)$$

In the present context only the instantaneous motion is of interest. As a measure of continued deformation, the rate of deformation tensor

$$\varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \qquad \ldots \quad (1.26)$$

is adopted. The vanishing of this tensor is a necessary and sufficient condition for instantaneous local rigidity.

Constitutive Equations

Viscoelastic materials show an increasing deformation under sustained loads, the rate of strain being dependant on the stress. These materials sometimes show a pronounced influence on the rate of loading, the strain being larger if the stress has grown more slowly to its final value. The constitutive equations of these materials may be linear or nonlinear. Here we mention only two fundamental models (i.e. the Maxwell and the Kelvin models) of linear viscoelastic materials.

The simplest Maxwell model (Figure 1.2)consists of a linear spring and a dashpot connected in series. The extensional strain ε_a of the spring follows the relation

$$\sigma = E\,\varepsilon_a \qquad \ldots \quad (1.27)$$

while the extensional strain ε_b of the dashpot obeys the law

$$\sigma = F\frac{\partial \varepsilon_b}{\partial t} = F\,\dot{\varepsilon}_b \qquad \ldots \quad (1.28)$$

where σ is the load per unit area, E and F are material constants , ε_a and ε_b are the strains in the spring and dashpot respectively, $\dot{\varepsilon}_b$ is the time derivative of the strain. Since, both elements are in series, the following relation is obtained

$$\frac{\dot{\sigma}}{E} + \frac{\sigma}{F} = \dot{\varepsilon}_a + \dot{\varepsilon}_b = \dot{\varepsilon}$$

or, alternatively,

$$\sigma + q_2 \dot{\sigma} = q_1 \dot{\varepsilon} \qquad \ldots \quad (1.29)$$

where,

$$q_2 = \frac{F}{E} \text{ and } q_1 = F .$$

The behaviour of this model is similar to that of a viscous fluid because the final value of stress tends to be zero with time.

As a second example,the Kelvin model for solids, in which the linear spring and dashpot are connected parallel (Figure 1.3), is considered. It should be understood that the stress σ is not to be distributed on the spring and dashpot following the law of levers, but at any time the strain of the two elements is the same and the total stress σ will be split into σ_a (spring) and σ_b (dashpot) in whichever way it is necessary to make ε the same. When applied to this model,

$$\sigma_a = E\,\varepsilon$$

and ,

$$\sigma_b = F\,\dot{\varepsilon}$$

Further, since $\sigma = \sigma_a + \sigma_b$,

$$\sigma = E\,\epsilon + F\,\dot{\epsilon}$$

or, in the standard form,

$$\sigma = (q_o + q_1 \frac{\partial}{\partial t})\ \epsilon \qquad \ldots (1.30)$$

where $q_o = E$ and $q_1 = F$.

The behaviour of this model is similar to that of viscous solid because stress tends to be finite, other than zero, with time. As such, normally this model is preferred to investigate the stability in viscoelastic materials.

The response of an element of viscoelastic solid to an infinitesimal change in the stress is assumed to be of the form

Change in stress = f (Change in strain)

or, in terms of stress-rate and strain-rate components,

Stress-rate component = f (Strain-rate components) ... (1.31)

It is desirable that the defining stress-rate be independent of any rigid body rotation imposed on the element. The Jaumann derivative of the Kirchhoff stress $\frac{\mathcal{D}\,\tau_{ij}}{\mathcal{D}\,t}$, which is associated with coordinate axes having the same spin as the material element and instantaneously coincident with the fixed coordinate

axes, is a suitable stress-rate. Consequently, following Hill (1959), (1.31) can be written as

$$\frac{D\tau_{ij}}{Dt} = K_{ijkl}\, e_{kl} \qquad \ldots \quad (1.32)$$

where the K_{ijkl} is the tensor of the viscoelastic moduli whose most general form is (Biot, 1965):

$$K_{ijkl} = Q_{ijkl} + \eta_{ijkl}\frac{\partial}{\partial t} + \int_0^t F_{ijkl}\,(t-\xi_1)\;d\xi_1 \qquad \ldots \quad (1.33)$$

subject to the requirement that

$$K_{ijkl} = K_{jikl} = K_{ijlk} = K_{klij} \qquad \ldots \quad (1.34)$$

If the incremental isotropy is preserved, and if the viscoelastic material is assumed to be incompressible and the material is supposed to possess the property of fading memory ($F_{ijkl} = 0$ in 1.33), the tensor K_{ijkl} can be shown to have the following form (Biot, 1965):

$$K_{ijkl} = (Q + \eta\frac{\partial}{\partial t})\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) + \bar{\lambda}\;\delta_{ij}\,\delta_{kl} \qquad \ldots \quad (1.35)$$

in which Q, η, $\bar{\lambda}$ are the material parameters which are functions of the current stress; and δ_{ij} is the Kronecker delta.

In view of (1.35), equation (1.32) can be rewritten as

$$\frac{\mathcal{D}\tau_{ij}}{\mathcal{D}t} = \left(2Q + 2\eta\frac{\partial}{\partial t}\right)\varepsilon_{ij} + \bar{\lambda}\,\varepsilon_{kk}\,\delta_{ij} \quad \ldots (1.36)$$

In the above, considering incompressibility, i.e.,

$$\varepsilon_{ii} \equiv v_{i,i} = 0 \quad \ldots (1.37)$$

the quantity $\bar{\lambda}\,\varepsilon_{kk}$ is indeterminate and equal to p, an arbitrary additive hydrostatic pressure. Hence, finally

$$\frac{\mathcal{D}\tau_{ij}}{\mathcal{D}t} = \left(2Q + 2\eta\frac{\partial}{\partial t}\right)\varepsilon_{ij} + p\,.\,\delta_{ij} \quad \ldots (1.38)$$

in which η is the viscosity coefficient.

It is worth mentioning here the connection between the nominal stress-rate $\dot{s}_{ij}$ and the Jaumann derivative of the Kirchhoff stress $\frac{\mathcal{D}\tau_{ij}}{\mathcal{D}t}$, which is as follows

$$\dot{s}_{ij} = \frac{\mathcal{D}\tau_{ij}}{\mathcal{D}t} + \sigma_{ik}\,.\,\omega_{jk} - \sigma_{jk}\,.\,\varepsilon_{ik} \quad \ldots (1.39)$$

where, $\varepsilon_{ij} = \frac{1}{2}\,(v_{i,j} + v_{j,i})$ is the strain-rate and is the symmetric part of the velocity gradient $v_{i,j}$.

The relation (1.39) is extremely important in view of the fact that the equations of motion (1.22) are expressed in terms of nominal stress-rate $\dot{s}_{ij}$. For the sake of convenient

reference in the following chapters, the constitutive equation (1.38) can now be expressed in terms of $\dot{s}_{ij}$ as follows:

$$\dot{s}_{ij} = (2Q + 2\eta \frac{\partial}{\partial t}) \varepsilon_{ij} + p\, \delta_{ij} + \sigma_{ik}\, \omega_{jk} - \sigma_{jk}\, \varepsilon_{ik} \qquad \ldots \quad (1.40)$$

It is to be noted that while $\frac{\mathcal{D} \tau_{ij}}{\mathcal{D} t}$ is a symmetric tensor, $\dot{s}_{ij}$ is not symmetric.

1.4 STABILITY CRITERION

Let the velocity v_i be prescribed on some part S_v of the surface S of the body and the dead loads be specified on the remaining surface S_T. The body which is in equilibrium, is imagined to be perturbed further by some kind of external agency. The current configuration of the body is said to be in a stable state if the magnitude of the ensuing displacement decays with time and is vanishingly small, whenever the perturbation itself is small. If, on the other hand, the amplitude remains finite for one type of disturbance, however small this might be or tends to increase with time, the state is said to be unstable. Therefore, in general, a sufficient condition for stability should be that the internal energy (the kinetic energy and the internal potential energy) stored should exceed the work done by the external loads during the resulting motion.

In the time interval δt, due to a virtual displacement $v_i \cdot \delta t$, the nominal stress changes from s_{ij} to $(s_{ij} + \dot{s}_{ij} . \delta t)$, still referred to the current state. Therefore, the increase in the energy will be,

$$\delta t \int_V (s_{ij} + \dot{s}_{ij} . \delta t) \, v_{j,i} \, dV \qquad \ldots \quad (1.41)$$

The kinetic energy stored is

$$\int_V \frac{1}{2} \rho \, v_j^2 \, dV = \frac{1}{2} \int_V \rho \, \dot{v}_j^2 \, (\delta t)^2 \, dV \qquad \ldots \quad (1.42)$$

The virtual work done by the external forces during the time interval δt, under the load on surface S_T and rigid constraints on surface S_V, i.e. $v_j \equiv 0$ on S_V, is

$$\delta t \int_S T_j \, v_j \, dS = \delta t \int_S n_i \, s_{ij} \, v_j \, dS = \delta t \int_V s_{ij} \, v_{j,i} dV$$

The total energy stored in the system minus the external work is

$$(\delta t)^2 \left[\int_V \frac{1}{2} \, \dot{s}_{ij} \, v_{j,i} \, dV + \int_V \frac{1}{2} \rho \, \dot{v}_j^2 \, dV \right]$$

which should be positive for the system to be stable ; hence, for stability,

$$\int_V \left[\frac{1}{2} \dot{s}_{ij} \, v_{j,i} + \frac{1}{2} \rho \, \dot{v}_j^2 \right] dV > 0 \qquad \ldots \quad (1.43)$$

for all continuous differentiable velocity fields which vanish on S_V (but are not identically zero) and are compatible through the constitutive equations with the stress rate $\dot{s}_{ij}$.

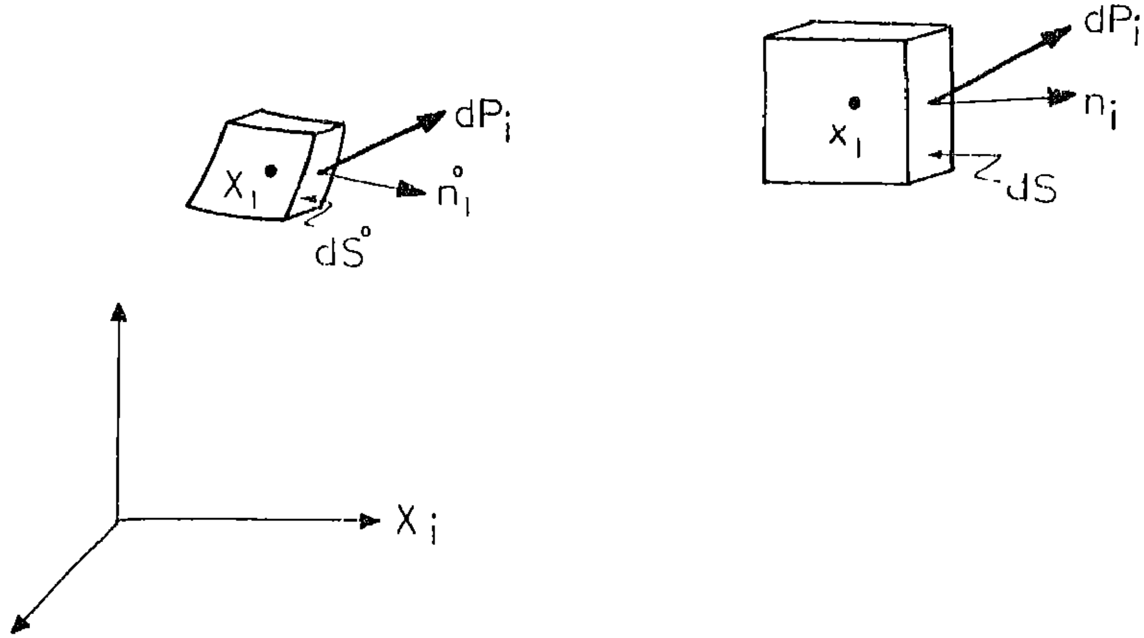


Fig.1·1 Concept of finite deformation

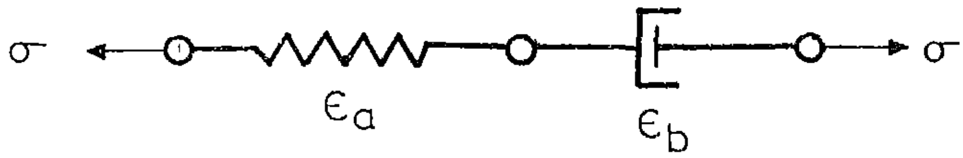


Fig.1·2 Spring and dashpot in series

Maxwell material

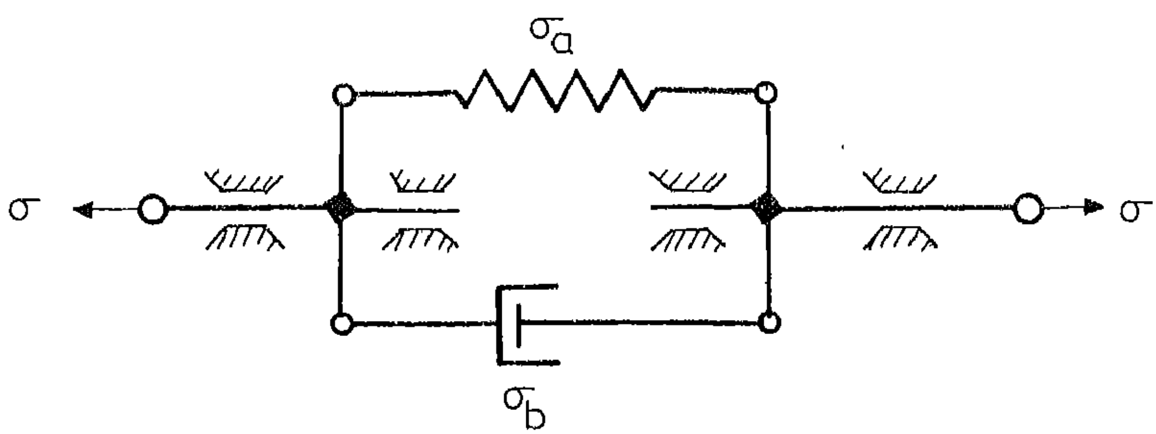


Fig.1·3 Spring and dashpot parallel

Kelvin material

CHAPTER : 2

CHAPTER : 2

INSTABILITIES IN RECTANGULAR VISCOELASTIC SOLIDS UNDER PLANE STRAIN

2.1 PROBLEM STATEMENT AND ITS FORMULATION

An incompressible viscoelastic rectangular solid undergoing continued deformation in plane strain is considered. The dimensions in the initial configuration are $2a_o \times 2b_o \times 2c_o$. It is brought to the current configuration by means of finite deformation (tension or compression) along one direction such that the dimension $2c_o$ of the specimen remains constant, i.e. the specimen undergoes deformation under plane strain. In the current configuration, the dimensions of the specimen are $2a \times 2b \times 2c_o$. Referred to this state, a fixed Cartesian coordinate system x_i, coinciding with the axes of symmetry of the specimen, is chosen as a reference frame. Whenever convenient, the coordinate x_i will be replaced by x,y,z and, the velocity components v_i by u,v,w, respectively. With respect to this frame, the deformation is assumed to be parallel to x-y plane, and the distribution of stress is assumed to be

$$\sigma_{ij} = \tau_{ij} = s_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma}{2} \end{bmatrix} \qquad \dots (2.1)$$

Under this state of stress the current configuration is in equilibrium. The aim is to find out under what conditions the current state is stable/unstable.

During perturbation from the current state, it is assumed that the longitudinal ends $x = \pm a$ are frictionless; hence the shear fraction rates on these ends will be zero. Also, the lateral faces $y = \pm b$ are supposed to be free of nominal traction rates. These conditions are expressed as,

$$\left.\begin{aligned} \dot{T}_2 &= n_i\, \dot{s}_{i2} = 0 \;, \quad x = \pm a \\ \dot{T}_j &= n_i\, \dot{s}_{ij} = 0 \;, \quad y = \pm b \end{aligned}\right\} \qquad \ldots (2.2)$$

where n_i is the unit outward normal to the boundary surface.

For the present case, the rate constitutive equation (1.40) yields following relations:

$$\left.\begin{aligned} \dot{s}_{11} &= \left(2Q + 2\eta \frac{\partial}{\partial t} - \sigma\right) v_{1,1} + p \\ \dot{s}_{22} &= \left(2Q + 2\eta \frac{\partial}{\partial t}\right) v_{2,2} + p \\ \dot{s}_{12} &= \left(Q + \eta \frac{\partial}{\partial t} - \frac{\sigma}{2}\right) v_{1,2} + \left(Q + \eta \frac{\partial}{\partial t} + \frac{\sigma}{2}\right) v_{2,1} \\ \dot{s}_{21} &= \left(Q + \eta \frac{\partial}{\partial t} - \frac{\sigma}{2}\right) v_{1,2} + \left(Q + \eta \frac{\partial}{\partial t} - \frac{\sigma}{2}\right) v_{2,1} \end{aligned}\right\} \qquad \ldots (2.3)$$

The equation of motion (1.22) now reduces to

$$\begin{aligned} \dot{s}_{11,1} + \dot{s}_{21,2} &= \rho \ddot{v}_1 \\ \dot{s}_{12,1} + \dot{s}_{22,2} &= \rho \ddot{v}_2 \end{aligned} \quad \ldots (2.4)$$

Finally, the incompressibility condition (1.37), for a plane strain problem may be written as

$$v_{1,1} + v_{2,2} = 0 \quad \ldots (2.5)$$

2.2 INSTABILITY CONDITIONS

Substitution of relations (2.3) in the equation of motion (2.4) yields the following two equations in the physical components of the velocity u, v and the parameter p

$$p_x + (2Q + 2\eta \frac{\partial}{\partial t} - \sigma)\, u_{xx} + (Q + \eta \frac{\partial}{\partial t} - \frac{\sigma}{2})\, u_{yy}$$

$$+ (Q + \eta \frac{\partial}{\partial t} - \frac{\sigma}{2})\, v_{xy} = \rho \ddot{u} \quad \ldots (2.6a)$$

$$p_y + (Q + \eta \frac{\partial}{\partial t} - \frac{\sigma}{2})\, u_{xy} + (Q + \eta \frac{\partial}{\partial t} + \frac{\sigma}{2})\, v_{xx}$$

$$+ (2Q + 2\eta \frac{\partial}{\partial t})\, v_{yy} = \rho \ddot{v} \quad \ldots (2.6b)$$

where, now a subscript denotes the partial differentiation. The boundary conditions (2.2) at $y = \pm b$ yield $\dot{s}_{22} = 0$ and $\dot{s}_{21} = 0$ i.e.

$$\left.\begin{aligned} (2Q + 2\eta \frac{\partial}{\partial t})\, v_y + p &= 0 \\ u_y + v_x &= 0 \end{aligned}\right\} \; y = \pm b \quad \ldots (2.7)$$

Let the velocity field and the parameter p be assumed in the following form

$$u = \Sigma \ f(y) \ \sin kx \exp(\bar{\omega}t)$$

$$v = \Sigma \ \varphi(y) \ \cos kx \exp(\bar{\omega}t) \qquad \ldots (2.8)$$

$$p = \Sigma \ -g(y) \ \cos kx \exp(\bar{\omega}t)$$

where $k = n\pi/a$ and $\bar{\omega} = i\,\omega$, $i = \sqrt{-1}$, ω being the frequency, $f(y)$, $\varphi(y)$ and $g(y)$ are functions of y to be determined. The assumed velocity distribution satisfies boundary conditions (2.2) at $x = \pm a$. Substitution of equations (2.8) in the incompressibility condition (2.5) and the equation of motion (2.6) yields three ordinary differential equations in $f(y)$, $\varphi(y)$ and $g(y)$:

$$kf + \varphi' = 0 \qquad \ldots (2.9a)$$

$$kg - \left(2Q + 2\eta\bar{\omega} - \sigma + \frac{\rho\bar{\omega}^2}{k^2}\right)k^2 f + \left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right)f''$$

$$-\left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right)k\varphi' = 0 \qquad \ldots (2.9b)$$

$$-g' - \left(Q + \eta\bar{\omega} + \frac{\sigma}{2} + \frac{\rho\bar{\omega}^2}{k^2}\right)k^2\varphi + \left(2Q + 2\eta\bar{\omega}\right)\varphi''$$

$$+ \left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right)kf' = 0 \qquad \ldots (2.9c)$$

where, now, the prime denotes differentiation with respect to y.

Eliminating f and g from the set of equations (2.9), the following fourth order equation in $\varphi(y)$ is obtained:

$$(Q + \eta\bar{\omega} - \frac{\sigma}{2})\varphi'''' - (2Q + 2\eta\bar{\omega} + \frac{\rho\bar{\omega}^2}{k^2})k^2\varphi''$$
$$+ (Q + \eta\bar{\omega} + \frac{\sigma}{2} + \frac{\rho\bar{\omega}^2}{k^2})k^4\varphi = 0$$

which, alternatively, can be written as

$$(1 + \lambda\bar{\omega} - \theta)\varphi'''' - (2 + 2\lambda\bar{\omega} + \frac{\alpha\bar{\omega}^2}{k^2})k^2\varphi''$$
$$+ (1 + \lambda\bar{\omega} + \theta + \frac{\alpha\bar{\omega}^2}{k^2})k^4\varphi = 0 \qquad \dots (2.10)$$

in which $\theta = \dfrac{\sigma}{2Q}$, $\lambda = \dfrac{\eta}{Q}$, $\dfrac{\rho}{Q} = \alpha$

With the help of equations (2.8), the boundary conditions (2.7) may be written as

$$\left.\begin{aligned} (2Q + 2\eta\bar{\omega})\varphi' - g &= 0 \\ f' - k\varphi &= 0 \end{aligned}\right\} y \pm b \qquad \dots (2.11)$$

which after eliminating the functions f and g, take the following forms

$$(3 + 3\lambda\bar{\omega} - \theta + \frac{\alpha\bar{\omega}^2}{k^2})\varphi' - (1 + \lambda\bar{\omega} - \theta)\frac{\varphi'''}{k^2} = 0 \quad \dots (2.11a)$$

$$\varphi'' + k^2\varphi = 0 \qquad \dots (2.11b)$$

such that $(1 + \lambda\bar{\omega} - \theta) \neq 0$.

Referring to the equation (2.10), let the solution of φ be taken in the following form

$$\varphi(y) = A \exp(\bar{\beta}' ky) \qquad \ldots (2.12)$$

where $k = n\pi/a$ as before.

The characteristic equation is

$$r_1 \bar{\beta}'^4 - r_2 \bar{\beta}'^2 + r_3 = 0 \qquad \ldots (2.13)$$

in which

$$r_1 = (1 + \lambda\bar{\omega} - \theta)$$

$$r_2 = (2 + 2\lambda\bar{\omega} + \frac{\alpha\bar{\omega}^2}{k^2})$$

$$r_3 = (1 + \lambda\bar{\omega} + \theta + \frac{\alpha\bar{\omega}^2}{k^2})$$

The roots $\bar{\beta}'_1$, $\bar{\beta}'_2$, $\bar{\beta}'_3$ and $\bar{\beta}'_4$ of the characteristic eq.(2.13) are

$$\bar{\beta}'_i = \pm \left[\frac{r_2 \pm (r_2^2 - 4 r_1 r_3)^{1/2}}{2r_1} \right]^{1/2} \qquad \ldots (2.14)$$

and are ordered such that $\bar{\beta}'_1 = -\bar{\beta}'_3$, $\bar{\beta}'_2 = -\bar{\beta}'_4$.

On simplification,

$$\bar{\beta}'_1 = -\bar{\beta}'_3 = 1, \text{ and}$$

$$\bar{\beta}'_2 = -\bar{\beta}'_4 = \left[\frac{1 + \lambda\bar{\omega} + \theta + \frac{\alpha\bar{\omega}^2}{k^2}}{1 + \lambda\bar{\omega} - \theta} \right]^{1/2} \qquad \ldots (2.15)$$

Two distinct cases arise, viz.,

i) Case of equal roots i.e. when $\bar{\beta}_1' = \bar{\beta}_2'$

ii) Case of unequal roots i.e. when $\bar{\beta}_1' \neq \bar{\beta}_2'$.

2.2.1 Case of Equal Roots

Equal roots are possible only if $\bar{\beta}_1' = \bar{\beta}_2' = -\bar{\beta}_3' = -\bar{\beta}_4' = 1$, i.e. when $r_2^2 = 4r_1 r_3$ in the expression of $\bar{\beta}_i'$ in equation (2.14). This leads to

$$\theta + \frac{\alpha\bar{\omega}^2}{2k^2} = 0$$

The solution of $\varphi(y)$ for this case is given by

$$\varphi(y) = A_1 \cosh ky + A_2 y \sinh ky + A_3 \sinh ky + A_4 y \cosh ky \qquad \dots (2.16)$$

where A_i ($i = 1,2,3,4$) are arbitary constants.

Substitution of the boundary conditions (2.11a and 2.11b) at $y = \pm b$ leads to four homogeneous, linear, algebraic equations for the constants A_i ($i = 1,2,3,4$). For a non-trivial solution to exist, the determinant of the co-efficients A_i should vanish ; this leads to

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{11} & a_{12} & -a_{13} & -a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ -a_{21} & -a_{22} & a_{23} & a_{24} \end{vmatrix} = 0 \qquad \dots (2.17)$$

where

$$a_{11} = 2 \cosh kb$$

$$a_{12} = \frac{2}{k} \cosh kb + 2b \sinh kb$$

$$a_{13} = 2 \sinh kb$$

$$a_{14} = \frac{2}{k} \sinh kb + 2b \cosh kb$$

$$a_{21} = (c_1 - c_2)\, k \sinh kb$$

$$a_{22} = (c_1 - 3c_2) \sinh kb + (c_1 - c_2)\, kb \cosh kb$$

$$a_{23} = (c_1 - c_2)\, k \cosh kb$$

$$a_{24} = (c_1 - 3c_2) \cosh kb + (c_1 - c_2)\, kb \sinh kb$$

in which

$$c_1 = 3 + 3\lambda\bar{\omega} - \theta + \frac{\alpha\,\bar{\omega}^2}{k^2}$$

$$c_2 = 1 + \lambda\bar{\omega} - \theta$$

The simplification of the condition (2.17) results in the following relation

$$\frac{(2kb)}{\sinh\,(2kb)} = 1 \qquad \ldots (2.18)$$

It can be easily shown that the condition (2.18) is satisfied if, and only if, $\frac{b}{a} \to 0$. Hence, instability is not a possibility in case of equal roots.

2.2.2 Case of Unequal Roots

In case of unequal roots (i.e. $\bar{\beta}_1 \neq \bar{\beta}_2$), the roots of the characteristic equation (2.13) being obtained from the expressions (2.14 or 2.15), the general solution of $\varphi(y)$ may be written as,

$$\varphi(y) = A_1 \cosh \bar{\beta}_1 ky + A_2 \cosh \bar{\beta}_2 ky + A_3 \sinh \bar{\beta}_1 ky + A_4 \sinh \bar{\beta}_2 ky \quad \ldots (2.19)$$

where, A_i (i= 1,2,3,4) are arbitary constants.

Substituting, $\bar{\beta}_1 = 1$, and ,

$$\beta = \bar{\beta}_2 = \left[\frac{1 + \lambda \bar{\omega} + \theta + \dfrac{\alpha \bar{\omega}^2}{k^2}}{1 + \lambda \bar{\omega} - \theta} \right]^{1/2} \quad \ldots (2.20)$$

equation (2.19) is rewritten as

$$\varphi(y) = A_1 \cosh ky + A_2 \cosh \beta ky + A_3 \sinh ky + A_4 \sinh \beta ky \quad \ldots (2.21)$$

Substitution of the boundary conditions (2.11a and 2.11b) at $y = \pm b$ leads to four homogeneous, linear, algebraic equations for the constants A_i (i = 1,2,3,4). For a non-trivial solution to exist, the determinant of the coefficients A_i should vanish as before. The condition for instability may thus be put in the following form:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ -a_{11} & -a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{21} & a_{22} & -a_{23} & -a_{24} \end{vmatrix} = 0 \qquad \ldots (2.22)$$

where, now,

$$a_{11} = c_1 \sinh kb \; ; \quad a_{21} = c_3 \cosh kb$$
$$a_{12} = c_2 \sinh \beta kb ; \quad a_{22} = c_4 \cosh \beta kb$$
$$a_{13} = c_1 \cosh kb \; ; \quad a_{23} = c_3 \sinh kb$$
$$a_{14} = c_2 \cosh \beta kb ; \quad a_{24} = c_4 \sinh \beta kb$$

in which

$$c_1 = \left(3 + 3\lambda\bar{\omega} - \theta + \frac{\alpha\,\bar{\omega}^2}{k^2}\right) - (1 + \lambda\bar{\omega} - \theta)$$
$$c_2 = \left(3 + 3\lambda\bar{\omega} - \theta + \frac{\alpha\,\bar{\omega}^2}{k^2}\right)\beta - (1 + \lambda\bar{\omega} - \theta)\,\beta^3$$
$$c_3 = 2 \;, \quad c_4 = (1 + \beta^2) \,.$$

The 4x4 determinant (2.22) can be decomposed into the product of two 2x2 determinants, viz.,

$$\Delta_1 \,.\, \Delta_2 = 0 \qquad \ldots (2.23)$$

where

$$\Delta_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \ldots \quad (2.24a)$$

and

$$\Delta_2 = \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \qquad \ldots \quad (2.24b)$$

Hence, for a non-trivial solution of A_i, equation (2.22) is satisfied if ,

either , $\Delta_1 = 0$ ... (2.25a)

or $\Delta_2 = 0$ ... (2.25b)

The above decomposition has a simple physical interpretation ; the conditions (2.25a) and (2.25b) correspond to instability phenomenon in the antisymmetric and symmetric modes, respectively.

For instability in the symmetric mode, $\Delta_2 = 0$ (cf.2.25b) leads to

$$\frac{\tanh(\beta\, kb)}{\tanh kb} = \frac{4\beta}{(1+\beta^2)^2} \qquad \ldots \quad (2.26a)$$

For instability in the antisymmetric mode, $\Delta_1 = 0$ (cf. 2.25a) reduces to

$$\frac{\tanh(\beta\, kb)}{\tanh kb} = \frac{(1+\beta^2)^2}{4\beta} \qquad \ldots \quad (2.26b)$$

where β is given by (2.20).

2.3 BEHAVIOUR OF THE INSTABILITY CONDITIONS

In addition to θ, (the non-dimensional stress parameter $\frac{\sigma}{2Q}$), the following two non-dimensional parameters are introduced:

$$\Omega = \left[\frac{\rho \ a^2}{Q} \right]^{1/2} \omega \quad ; \quad \lambda_1 = \left[\frac{1}{\rho \ Q \ a^2} \right]^{1/2} \eta \qquad \ldots \ (2.27)$$

These two parameters are referred to as non-dimensional frequency and non-dimensional viscosity in the subsequent discussions. Therefore, the instability conditions (2.26a and 2.26b), in terms of these parameters, will read as

$$\frac{\tanh \beta \ kb}{\tanh kb} = \frac{4\beta}{(1 + \beta^2)^2} \qquad \ldots \ (2.28a)$$

and ,

$$\frac{\tanh \beta \ kb}{\tanh kb} = \frac{(1 + \beta^2)^2}{4\beta} \qquad \ldots \ (2.28b)$$

where, now

$$\beta = \left[\frac{1 + i \lambda_1 \Omega + \theta - \dfrac{\Omega^2}{n^2 \pi^2}}{1 + i \lambda_1 \Omega - \theta} \right]^{1/2}$$

Numerical solution of the equations (2.28a and 2.28b) for a particular value of kb (i.e. for a given aspect ratio b/a), will yield various values of β (cf. Figs. 2.1, 2.2, 2.3, 2.4, 2.5). Let the numerical value of β obtained by solving either of the equations (2.28a) or (2.28b), for a particular value of kb, be designated by $\bar{\beta}$. The value of $\bar{\beta}$

will either be real or imaginary. Actual numerical computations did not yield any complex value of $\bar{\beta}$. Again, it is evident from the figures (Figs. 2.1, 2.2, 2.3, 2.4, 2.5) that

(i) if $\bar{\beta}$ is real $0 \leq |\bar{\beta}| \leq 1$ and

(ii) if $\bar{\beta}$ is imaginary $i \leq |\bar{\beta}| \leq \gamma i$

where γ is real number and is ≥ 1.

Once, a particular value of $\bar{\beta}$ has been determined (for a particular aspect ratio), load (stress) vs. frequency relationship may be studied from the relation

$$\bar{\beta} = \left[\frac{1 + i\lambda_1 \Omega + \theta - \dfrac{\Omega^2}{n^2\pi^2}}{1 + i\lambda_1 \Omega - \theta} \right]^{1/2} \quad \ldots (2.29)$$

In general, the non-dimensional frequency Ω is complex ; $\bar{\beta}$ which satisfies either of the relations (2.28a and 2.28b) may be real or imaginary ; non-dimensional load θ is real, either positive (tensile) or negative (compressive). Equation (2.29) on simplification yields the following equation in non-dimensional frequency Ω :

$$\left(\frac{\Omega}{n\pi} \right)^2 - \left[i(1 - \bar{\beta}^2) \lambda_1 \, n\pi \right] \left(\frac{\Omega}{n\pi} \right) - \left[(1-\theta)\,(1-\bar{\beta}^2) + 2\theta \right] = 0$$

from which,

$$\frac{\Omega}{n\pi} = i\left[(1-\bar{\beta}^2)\frac{\lambda_1 n\pi}{2}\right] \pm \left[\left\{i(1-\bar{\beta}^2)\left(\frac{\lambda_1 n\pi}{2}\right)\right\}^2 + \left\{(1-\theta)(1-\bar{\beta}^2) + 2\theta\right\}\right]^{1/2} \qquad \ldots (2.30)$$

The following observations can now be made:

i) As $\bar{\beta}$ is real or imaginary (but never complex), the second term following the $\pm$ sign within the square bracket in equation (2.30) is a real number, positive or negative.
Further, if the second term within the square bracket is positive it represents the real part of the non-dimensional frequency.

ii) The first term within the square bracket in equation (2.30) is always positive, since, $0 \leq |\bar{\beta}| \leq 1$, when $\bar{\beta}$ is real and $i \leq |\bar{\beta}| \leq \gamma i$, when $\bar{\beta}$ is imaginary, γ is a real number ≥ 1.

iii) When the second term within the square bracket is real as well as positive, the first term within the square bracket in equation (2.30) will represent the imaginary part of the non-dimensional frequency. Hence, if the second expression within the square bracket is positive, this will lead to oscillatory convergence of the system, the non-dimensional frequency being complex.

iv) If the second term within the square bracket is negative (i.e. the second term itself is imaginary), but the second expression itself is less than the first term (which is positive imaginary), the non-dimensional frequency in equation (2.30) will be imaginary but positive. This will lead to exponential convergence of the system.

v) Hence, for instability (i.e. divergence), the second expression within the square bracket in equation (2.30) should be negative, as well as,

magnitude of the second expression

$\geq$ magnitude of the first expression in equation (2.30).

Equation (2.30) is rearranged in the following form

$$\left(\frac{\Omega}{n\pi}\right) = i\left[(1-\bar{\beta}^2)\left(\frac{\lambda_1 n\pi}{2}\right)\right]$$
$$\pm i\left[\left\{(1-\bar{\beta}^2)\left(\frac{\lambda_1 n\pi}{2}\right)\right\}^2 - \left\{(1-\theta)(1-\bar{\beta}^2) + 2\theta\right\}\right]^{1/2} \quad \ldots (2.31)$$

Hence, the following should be satisfied for divergence,

$$\left[\left\{(1-\bar{\beta}^2)\left(\frac{\lambda_1 n\pi}{2}\right)\right\}^2 - \left\{(1-\theta)(1-\bar{\beta}^2) + 2\theta\right\}\right]^{1/2}$$
$$\geq \left[(1-\bar{\beta}^2)\left(\frac{\lambda_1 n\pi}{2}\right)\right] \quad \ldots (2.32)$$

which on simplification, reduces to

$$\theta \leq -\left(\frac{1-\bar{\beta}^2}{1+\bar{\beta}^2}\right) \qquad \ldots (2.33)$$

or

$$\theta \leq \left(\frac{\bar{\beta}^2-1}{\bar{\beta}^2+1}\right) \qquad \ldots (2.33a)$$

From the requirement (2.33), it is evident that for instability in compression (i.e. θ negative), in either symmetric or antisymmetric mode, modulus of $\bar{\beta}$ (whether $\bar{\beta}$ is real or imaginary) should lie between zero and one. Further, the instability in tension is not a possibility.

2.3.1 Instability in Compression

Case (i) Symmetric Mode : Real Roots:

Referring to the equation (2.28a) and Fig. 2.1, $\bar{\beta} = 0$ or ± 1 always for any value of kb. If kb is sufficiently large (i.e. if b/a is large enough so that kb > 4), another real value of $\bar{\beta}$ is possible, the absolute value of which lies between zero and 0.29561 (cf.Fig. 2.2). The value $\bar{\beta} = \pm 1$ need not be considered, since, as has already been explained ,there can be no instability in the case of equal roots. The case $\bar{\beta} = 0$ leads to (cf. equation 2.33)

$$\theta \leq -1.0 \quad , \quad 0 \leq kb \leq 4.0$$

When kb > 4, the root, absolute value of which lies between 0 and 0.29561 (cf. Fig.2.2) will yield a lower value of θ than above.

When $\frac{b}{a} \rightarrow \infty$, i.e. $kb \rightarrow \infty$, equation (2.28a) reduces to

$$\frac{4\bar{\beta}}{(1+\bar{\beta}^2)^2} = 1$$

Therefore, $\bar{\beta} = \pm 0.29561$ and hence

$$\theta \leq -0.83927$$

when $b/a \rightarrow \infty$.

Further, as $\frac{b}{a} \rightarrow 0$, i.e. $kb \rightarrow 0$, equation (2.28a) yields

$$\beta = \frac{4\beta}{(1+\beta^2)^2}$$

Solving, $\bar{\beta} = 0$ and ± 1, are the only possible real roots, which yield $\theta = -1.0$.

Therefore, for instability in the symmetric mode under compression

$$-1.0 \leq \theta \leq -0.83927, \quad \text{when} \quad 4 \leq kb < \infty, \quad \text{and}$$
$$\theta = -1.0, \quad \text{when} \quad 0 \leq kb \leq 4.0 \qquad \ldots (2.34)$$

Case (ii) Symmetric Mode: Imaginary Roots:

When β is imaginary (let $\beta = \gamma i$, where γ is real), equation (2.28a) reduces to

$$\frac{\tan(\gamma\, kb)}{\tanh kb} = \frac{4\gamma}{(1-\gamma^2)^2} \qquad \ldots (2.35)$$

Solution of the above equation yields infinite number of imaginary roots (ref. Fig. 2.4) and equation (2.33), taking $\bar{\beta} = \bar{\gamma} i$ takes the following form

$$\theta \leq -\left(\frac{1+\bar{\gamma}^2}{1-\bar{\gamma}^2}\right) \qquad \ldots (2.36)$$

It is evident that the least value of θ is -1.0 when $\bar{\gamma} = 0$, which is always a solution of equation (2.35) regardless of the value of kb or b/a. Also, for θ to be negative, $0 \leq |\bar{\gamma}| \leq 1.0$.

Considering both the real and imaginary roots of β, modulus of which lies between zero and one, the value of θ may be stated to lie as follows

$$\bar{\beta} = 0 \quad , \quad \theta = -1.0 \quad , \quad 0 \leq kb \leq 4.0$$

$$0 \leq |\bar{\beta}| \leq 0.29561, \quad -1.0 \leq \theta \leq -0.83927; \quad 4 \leq kb < \infty \qquad \ldots (2.37)$$

<u>Case (iii) Antisymmetric Mode: Real Root:</u>

Referring to the equation (2.28b) and Fig. 2.3, $\bar{\beta} = \pm 1$ and another pair of roots $\bar{\beta}$, such that $0.29561 \leq |\bar{\beta}| \leq 1.0$, when $0 \leq kb < \infty$. As before, $\bar{\beta} = \pm 1$ are not considered because instability cannot occur when the roots are equal. Thus, for any real value of β, $0 \leq |\bar{\beta}| \leq 1.0$,

$$-0.83927 \leq \theta \leq 0 \; ; \quad 0 \leq kb < \infty \qquad \ldots (2.38)$$

As $\frac{b}{a} \rightarrow 0$ i.e. $kb \rightarrow 0$ and hence, the equation (2.28b) reduces to

$$\beta = \frac{(1+\beta^2)^2}{4\beta}$$

whose solutions are

$$\bar{\beta} = \pm 1$$

Again as $\frac{b}{a} \rightarrow \infty$, i.e. $kb \rightarrow \infty$, and the equation (2.28b) becomes

$$1 = \frac{(1+\beta^2)^2}{4\beta}$$

whose solutions are

$\bar{\beta} = \pm 0.29561$, and hence from equation (2.33)

$\theta = - 0.83927$

Case (iv) Antisymmetric Mode: Imaginary Roots:

When β is imaginary (let $\beta = \gamma i$, γ is real as before), equation (2.28b) reduces to

$$\frac{\tan(\gamma\, kb)}{\tanh kb} = - \frac{(1-\gamma^2)^2}{4\gamma} \qquad \ldots (2.39)$$

Solution of the above equation yields infinite number of imaginary roots (ref. Fig. 2.5), but always $|\bar{\gamma}| \geq 1.0$, i.e. for practical purpose, no imaginary roots are to be considered in order to determine the

non-dimensional compressive load.

As $\frac{b}{a} \rightarrow 0$ i.e. $kb \rightarrow 0$ the equation (2.39) takes the following form

$$\gamma = - \frac{(1-\gamma^2)^2}{4\gamma} \qquad \ldots (2.40)$$

which does not yield any real value of γ.

As $\frac{b}{a} \rightarrow \infty$ i.e. $kb \rightarrow \infty$, the equation (2.39) reduces to

$$\tan (\gamma\, kb) = - \frac{(1-\gamma^2)^2}{4\gamma} \qquad \ldots (2.41)$$

Solution of this equation (2.41) yields infinite number of roots modulus of which are greater than 1, hence need not be considered for the compression case.

Therefore considering expressions (2.38, 2.40, 2.41) instability in compression can occur, if

$$0.29561 \leq |\bar{\beta}| \leq 1.0 \; ; \; - 0.83927 \leq \theta \leq 0 \; ; \; 0 \leq kb < \infty \qquad \ldots (2.42)$$

Comparing expressions (2.37 and 2.42), it is observed that the instability in the antisymmetric mode is the only possibility.

2.4 NUMERICAL RESULTS AND DISCUSSIONS

The aim is to find out the critical value of β (real and/or imaginary) from the instability conditions (2.28a and 2.28b) and hence determine from the equation (2.33) the non-dimensional load θ at which the material becomes unstable. The computations were performed for all possible dimensions of b/a. Three values of viscosity λ_1 (0.01, 0.1, 0.5) were taken. The observations are as follows:

i) The critical value of the load θ is independent of the magnitude of non-dimensional viscosity, and the nature of motion is divergent. So long as the load does not exceed the limit prescribed by equation(2.33) nature of motion will be either oscillatory or exponential. As soon as the load exceeds the limit prescribed by equation (2.33), motion diverges exponentially. Right at the critical value of the load, at least one value of the frequency Ω becomes zero i.e. real as well as imaginary part of the frequency becomes zero. Although the cricical load at instability is independent of non-dimensional viscosity λ_1, the rate at which the motion decays, when the load does not exceed the limit specified by equation (2.33), as well as the rate at which the motion diverges,when the load exceeds the limit specified by equation (2.33), depend on the magnitude of the viscosity λ_1. This has been verified numerically.

ii) For a rectangular viscoelastic solid, no loss of stability is possible in tension.

iii) The primary mode of instability in a rectangular viscoelastic solid under compression in plain strain, is antisymmetric. This means that the condition (2.28b) yields the lower value of compressive load/stress (at which the motion becomes unstable) than that obtained from the condition (2.28a). This is also evident from the plot of nondimensional load/stress θ versus specimen dimensions (b/a or a/b) in Fig. 2.6.

Further, when $\frac{b}{a}$ is large ($\frac{a}{b} = 0$), both symmetric and antisymmetric modes of instability are possible under compression giving identical values of load/stress ($\theta = -0.83927$).

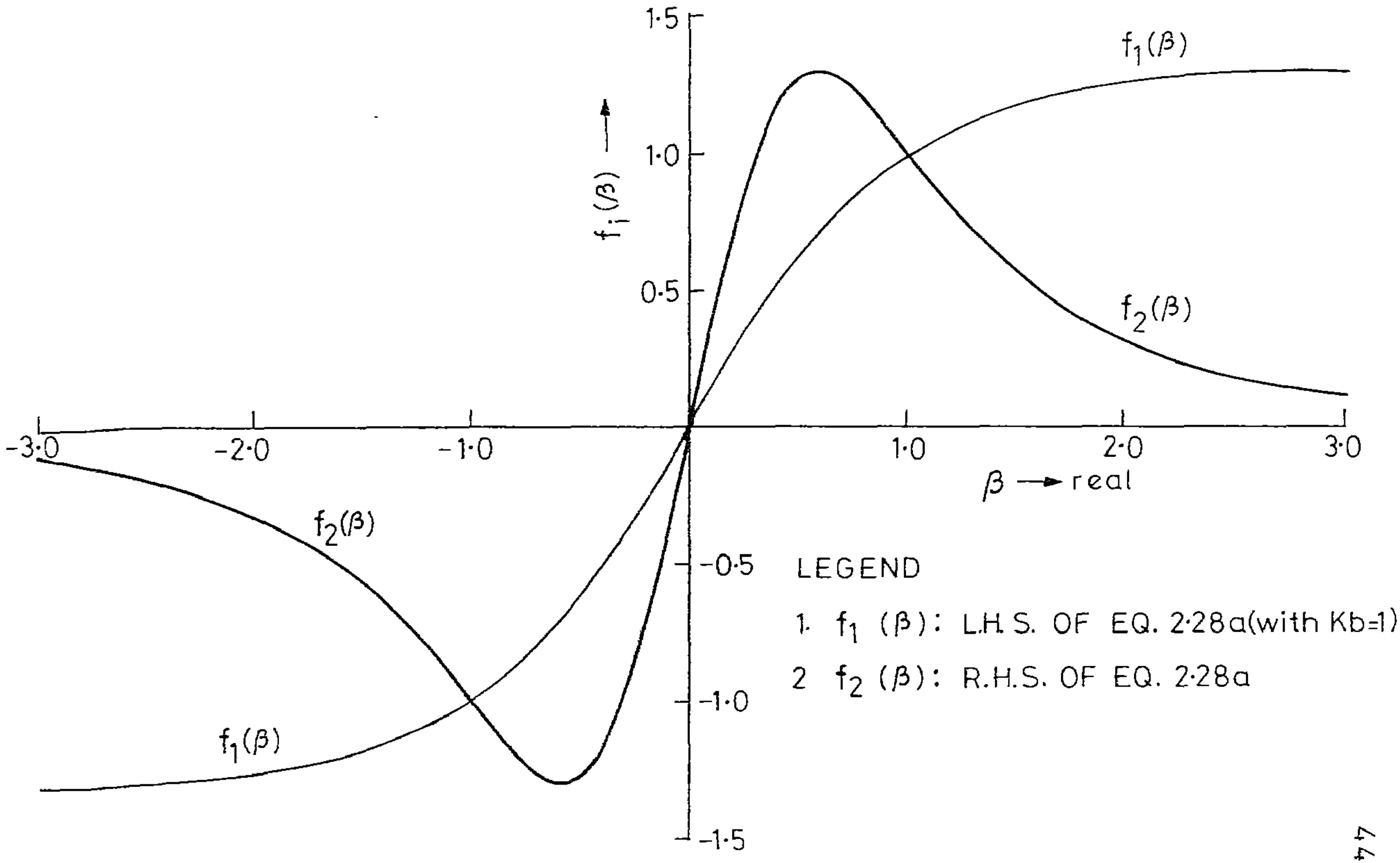


Fig 2·1 Distribution of real roots in symmetric mode ($0 \leq Kb < 4$)

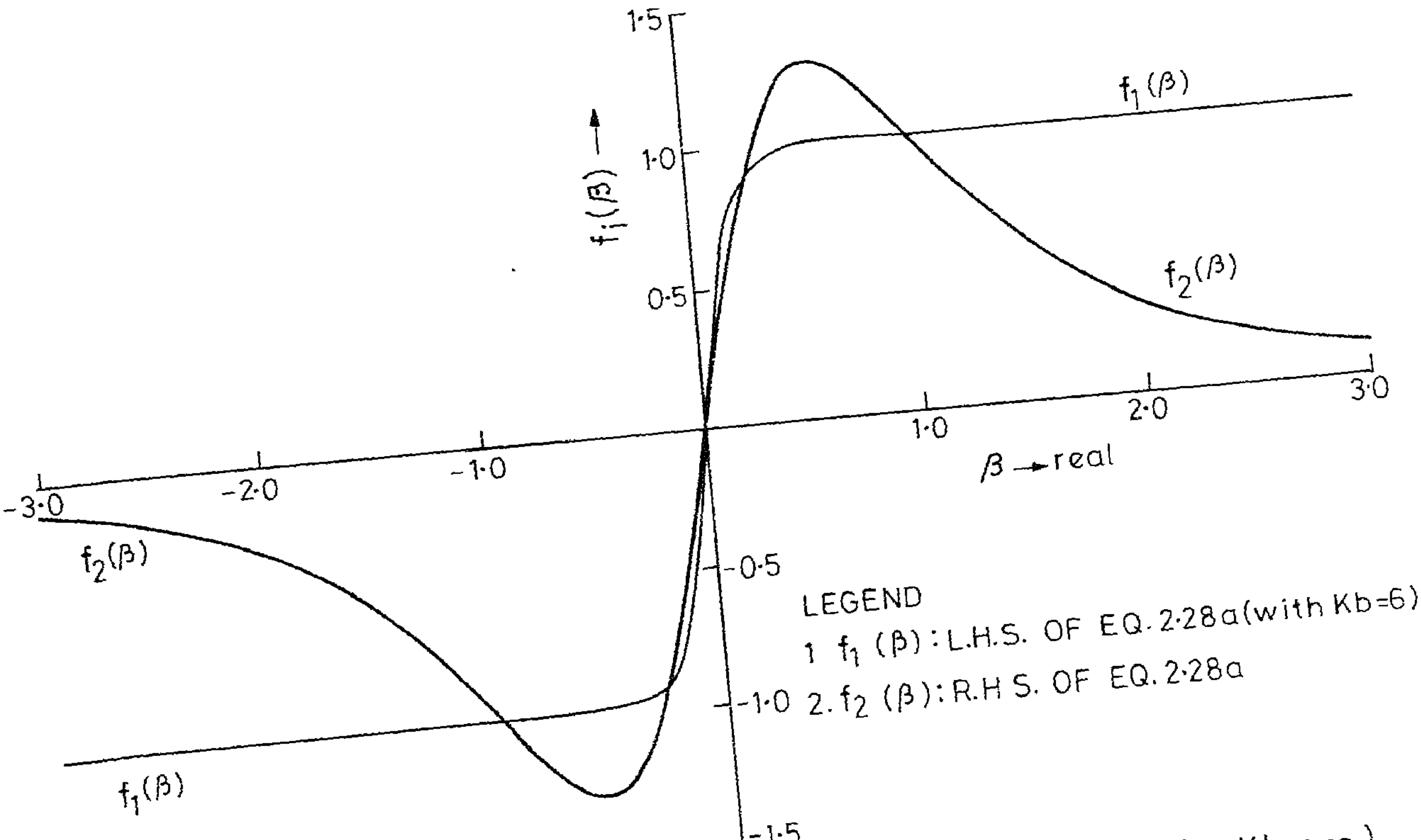


Fig. 2·2 Distribution of real roots in symmetric mode ($4 < Kb < \infty$)

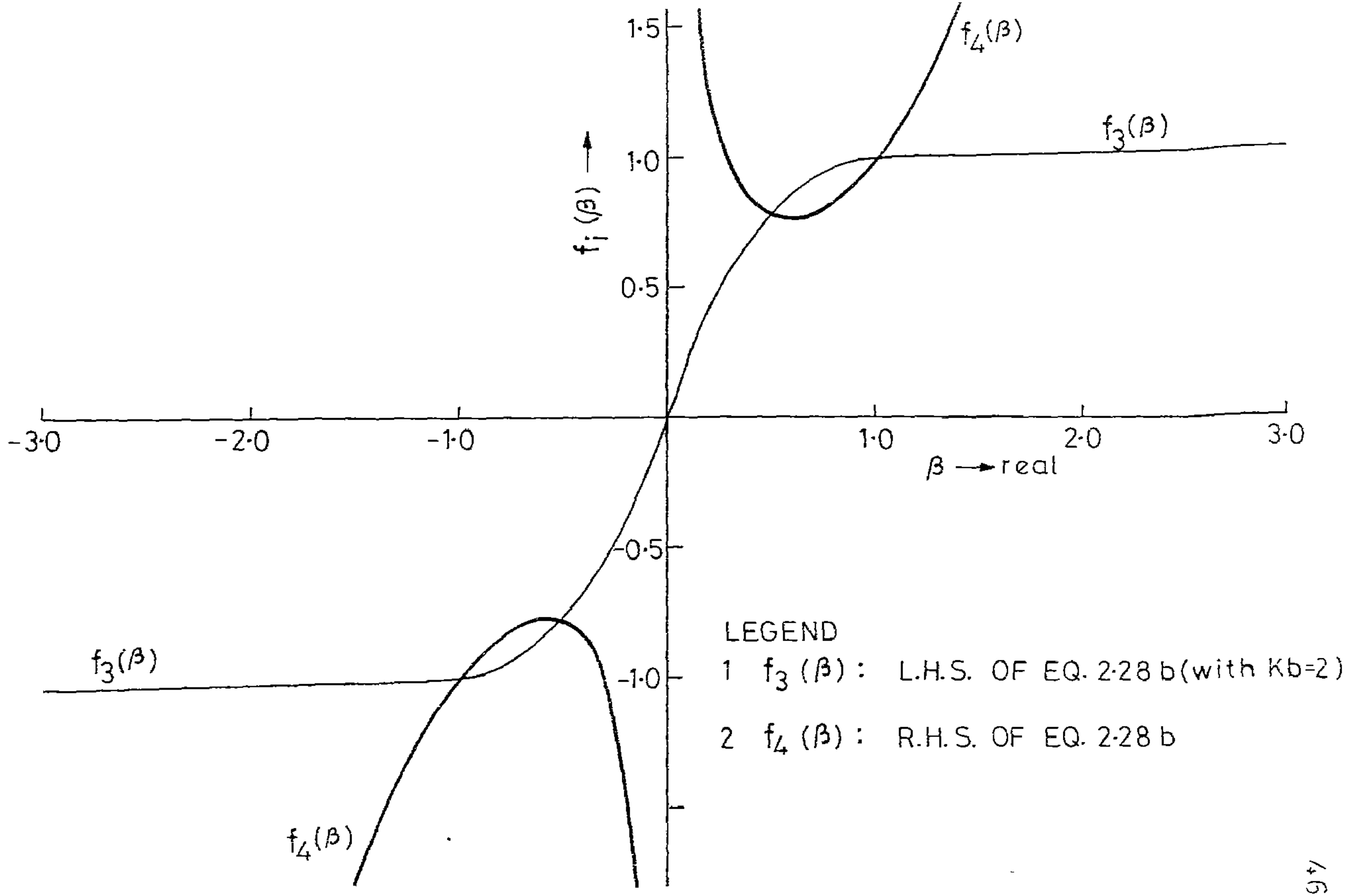
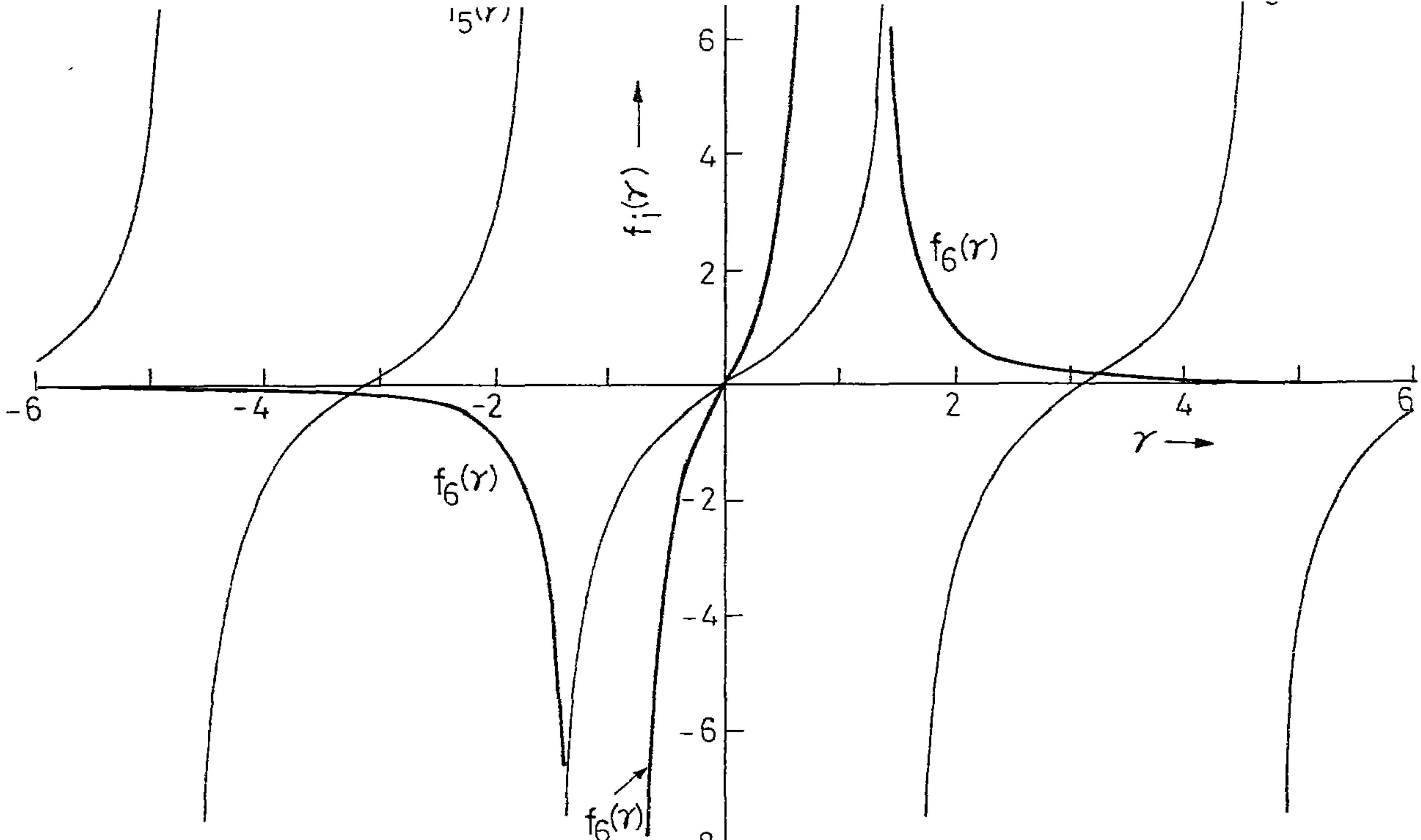


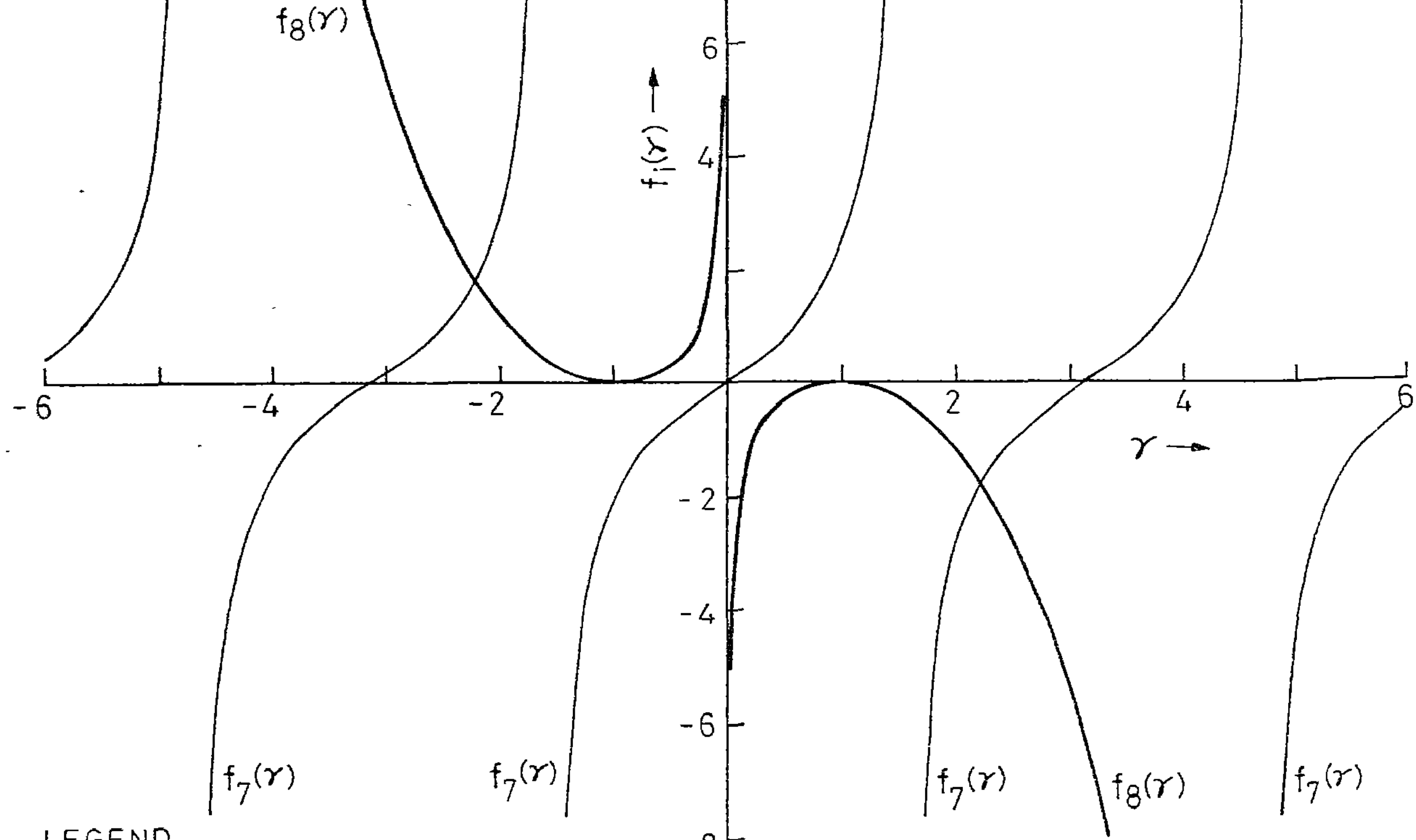
Fig. 2·3 Distribution of real roots in antisymmetric mode



LEGEND

1. f_5 (γ) : L H.S. OF EQ. 2·35 (with Kb = 1)
2. f_6 (γ) : R.H.S. OF EQ. 2·35

Fig. 2·4 Distribution of imaginary roots in symmetric mode



LEGEND

1. $f_7(\gamma)$: L.H.S. OF EQ. 2·39 (with Kb=1)
2. $f_8(\gamma)$: R.H.S OF EQ. 2·39

Fig. 2·5 Distribution of imaginary roots in antisymmetric mode

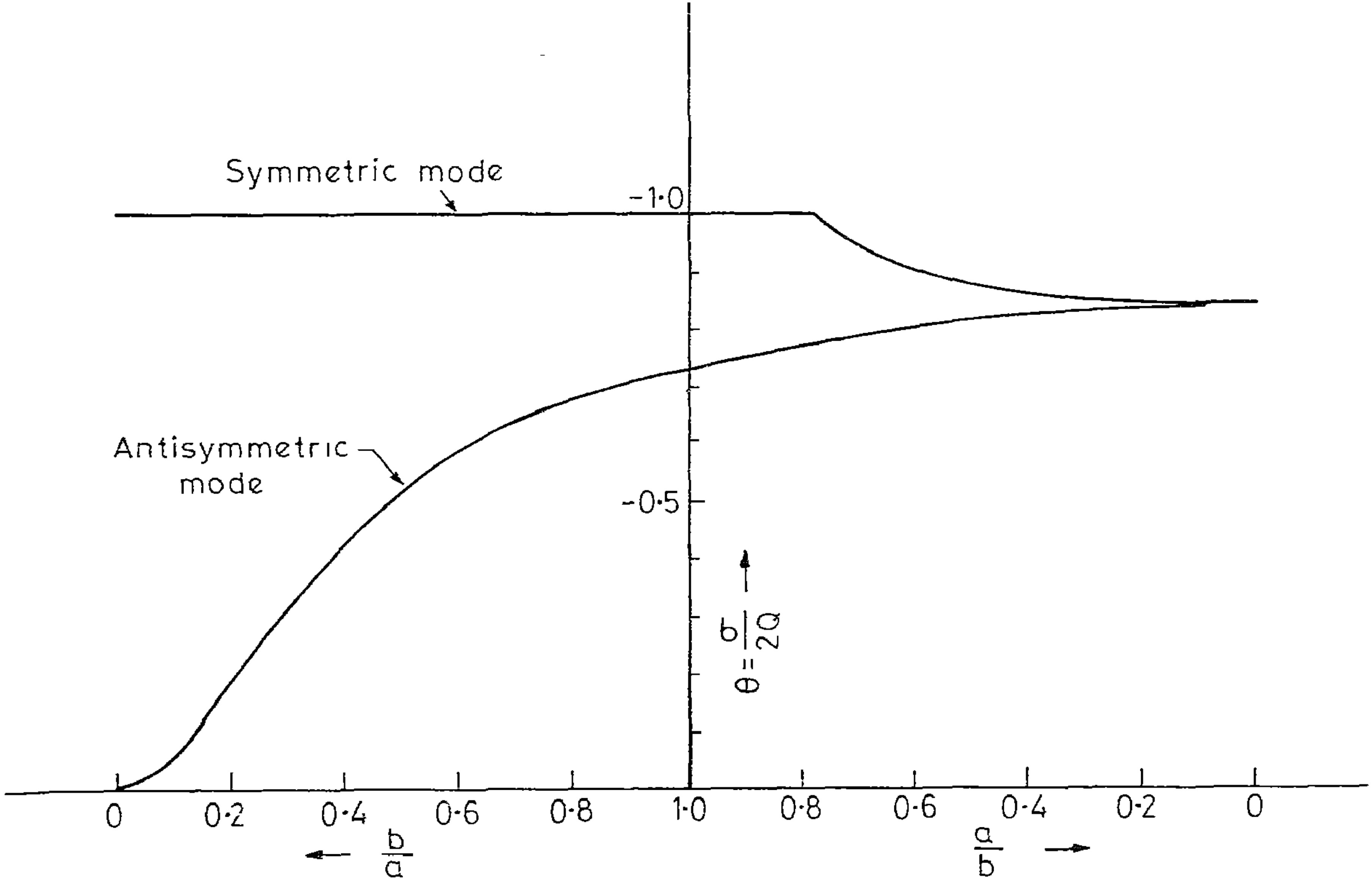


Fig. 2·6 Variation of critical stress in compression with aspect ratio in symmetric and antisymmetric modes

CHAPTER : 3

CHAPTER : 3

AXISYMMETRIC INSTABILITIES IN VISCOELASTIC CIRCULAR CYLINDER UNDER AXIAL LOAD

3.1 PROBLEM STATEMENT AND ITS FORMULATION

In this chapter, the stability of a viscoelastic solid circular cylinder under axial load has been examined. At the current instant, during the process of continuing deformation, the cylinder has radius a and length 2L. To investigate the instability phenomenon, the behaviour of the cylinder in transition from the current state to a neighbouring state, under infinetesimal increments of boundary values, has been isolated for study. Referred to the current state, a fixed cylindrical coordinate system x_i, with one of the axes coinciding with the axis of symmetry of the cylinder has been taken as the reference frame. Whenever convenient, the co-ordinates x_1, x_2, x_3 will be replaced by r, θ, z, respectively, and the velocity components v_1, v_2, v_3 by u, v, w, respectively.

In the current state, the internal stress distribution, assumed homogeneous, is supposed to be given by

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma \end{bmatrix} \qquad \ldots \quad (3.1)$$

where σ is the stress in the axial direction.

During the incremental deformation from the current state of equilibrium, the lateral cylindrical surfaces are assumed to be free of any traction-rates while the cylinder ends are frictionless (i.e. shear traction-rates are zero on these two ends), and these two ends move with constant velocity relative to each other. Therefore, the boundary conditions for the non-homogeneous incremental deformation are

$$\left.\begin{aligned} v_3 &= 0 \\ \dot{T}_j &= n_i\, \dot{s}_{ij} = 0\ , \ j = 1,2 \end{aligned}\right\} \ z = \pm L \qquad \ldots \quad (3.2)$$

$$\dot{T}_j = n_i\, \dot{s}_{ij} = 0\ , \quad r = a \qquad \ldots \quad (3.3)$$

and, $$\dot{T}_j = n_i\, \dot{s}_{ij} \text{ are finite at } r = 0 \qquad \ldots \quad (3.4)$$

where n_i is the unit outward normal to the boundary surface. Alternatively, these boundary conditions may be expressed as follows

$$\left.\begin{aligned} w &= 0 \\ \dot{s}_{zr} &= \dot{s}_{z\theta} = 0 \end{aligned} \quad z = \pm L \right\} \qquad \ldots \quad (3.2a)$$

$$\dot{s}_{rr} = \dot{s}_{r\theta} = \dot{s}_{rz} = 0\ , \quad r = a \qquad \ldots \quad (3.3a)$$

and , $$\dot{s}_{rr}\ ,\ \dot{s}_{r\theta}\ ,\ \dot{s}_{rz} \text{ are finite at } r = 0 \qquad \ldots \quad (3.4a)$$

For axisymmetric case, $u = u(r,z)$, $v = 0$, $w = w(r,z)$, and the equation of motion (1.22) assumes the following form in the present case

$$\left.\begin{aligned} \dot{s}_{rr,r} + \frac{1}{r}(\dot{s}_{rr} - \dot{s}_{\theta\theta}) + \dot{s}_{zr,z} &= \rho\ddot{u} \\ \dot{s}_{rz,r} + \frac{1}{r}\dot{s}_{rz} + \dot{s}_{zz,z} &= \rho\ddot{w} \end{aligned}\right\} \quad \ldots \quad (3.5)$$

where, as before, ρ is the current density of the material, and a comma indicates partial differentiation.

Furthermore, the condition of incompressibility (1.37) reduces to

$$u_{,r} + \frac{u}{r} + w_{,z} = 0 \qquad \ldots \quad (3.6)$$

For the axisymmetric case considered here, the components of the strain-rate tensor $\varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ and those of the rotation tensor $\omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ are given as follows

$$\varepsilon_{ij} = \begin{bmatrix} \frac{\partial u}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \\ 0 & \frac{u}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) & 0 & \frac{\partial w}{\partial z} \end{bmatrix} \qquad \ldots \quad (3.7)$$

$$\omega_{ij} = \begin{bmatrix} 0 & 0 & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) \\ 0 & 0 & 0 \\ -\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) & 0 & 0 \end{bmatrix} \qquad \ldots (3.8)$$

With the help of the relations (3.7) and (3.8), the constitutive equation (1.40) yields following relations

$$\begin{aligned}
\dot{s}_{rr} &= (2Q + 2\eta D)\, u_{,r} + p \\
\dot{s}_{\theta\theta} &= (2Q + 2\eta\, D)\, \frac{u}{r} + p \\
\dot{s}_{zz} &= (2Q + 2\eta\, D - \sigma)\, w_{,z} + p \qquad \ldots \quad (3.9) \\
\dot{s}_{rz} &= (Q + \eta\, D - \frac{\sigma}{2})\, (u_{,z} + w_{,r}) \\
\dot{s}_{zr} &= (Q + \eta\, D + \frac{\sigma}{2})\, u_{,z} + (Q + \eta\, D - \frac{\sigma}{2})\, w_{,r} \\
\dot{s}_{r\theta} &= \dot{s}_{\theta r} = \dot{s}_{\theta z} = \dot{s}_{z\theta} = 0
\end{aligned}$$

where D stands for the operator $\frac{\partial}{\partial t}$.

3.2 INSTABILITY CONDITIONS

Substitution of the relations (3.9) in the equation of motion (3.5) results in the following two equations:

$$\left.\begin{aligned} &(2Q + 2\eta D)\, u_{,rr} + p_{,r} + \frac{1}{r}(2Q + 2\eta D)\left(u_{,r} - \frac{u}{r}\right) \\ &\quad + \left(Q + \eta D + \frac{\sigma}{2}\right) u_{,zz} + \left(Q + \eta D - \frac{\sigma}{2}\right) w_{,rz} = \rho\, \ddot{u} \\ &\text{and,} \\ &\left(Q + \eta D - \frac{\sigma}{2}\right)(u_{,zr} + w_{,rr}) + \frac{1}{r}\left(Q + \eta D - \frac{\sigma}{2}\right)(u_{,z} + w_{,r}) \\ &\quad + (2Q + 2\eta D - \sigma)\, w_{,zz} + p_{,z} = \rho\, \ddot{w} \end{aligned}\right\} \quad \ldots (3.10)$$

In view of the boundary conditions (3.2a), the solution of equations (3.10) may be taken in the following form

$$\left.\begin{aligned} u &= f_1(r) \cos kx \exp(\bar{\omega} t) \\ w &= f_2(r) \sin kx \exp(\bar{\omega} t) \\ p &= f_3(r) \cos kx \exp(\bar{\omega} t) \end{aligned}\right\} \quad \ldots (3.11)$$

in which $k = \frac{n\pi}{L}$, n being an integer, $\bar{\omega} = i\omega$,

ω being the frequency, $i = \sqrt{-1}$, and $f_i(r)$ are unknown functions to be determined.

Use of (3.11) in (3.10) and in the incompressibility condition (3.6) yields the following three equations in f_1, f_2 and f_3:

$$f_3' - \left(Q + \eta\bar{\omega} + \frac{\sigma}{2} + \frac{\rho\,\bar{\omega}^2}{k^2}\right) k^2 f_1 - \frac{1}{r^2}(2Q + 2\eta\bar{\omega})\, f_1$$
$$+ \frac{1}{r}(2Q + 2\eta\bar{\omega})\, f_1' + (2Q + 2\eta\bar{\omega})\, f_1'' + \left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right) k f_2'$$
$$= 0 \qquad \ldots (3.12a)$$

$$- k f_3 - \left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right) k f_1' - \frac{1}{r}\left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right) k f_1$$
$$+ \left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right) f_2'' + \frac{1}{r}\left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right) f_2'$$
$$- \left(2Q + 2\eta\bar{\omega} - \sigma + \frac{\rho\,\bar{\omega}^2}{k^2}\right) k^2 f_2 = 0 \qquad \ldots (3.12b)$$

$$f_1' + \frac{f_1}{r} + k f_2 = 0 \qquad \ldots (3.12c)$$

in which, now, the prime denotes differentiation w.r.t. r.

Eliminating f_2 and f_3 in the above equations, the resulting equation in $f_1(r)$ is arranged as follows:

$$f_1'''' + a_1 f_1''' + a_2 f_1'' + a_3 f_1' + a_4 f_1 = 0 \qquad \ldots (3.13)$$

where,

$$a_1 = \frac{2}{r}$$

$$a_2 = 2k^2 \left[- \frac{\left(1 + \lambda\bar{\omega} + \frac{\alpha\bar{\omega}^2}{2k^2}\right)}{(1 + \lambda\bar{\omega} - \theta)} \right] - \frac{3}{r^2}$$

$$a_3 = \frac{2k^2}{r} \left[- \frac{\left(1 + \lambda\bar{\omega} + \frac{\alpha\bar{\omega}^2}{2k^2}\right)}{(1 + \lambda\bar{\omega} - \theta)} \right] + \frac{3}{r^3}$$

$$a_4 = k^4 \left[\frac{\left(1 + \lambda\bar{\omega} + \theta + \frac{\alpha\bar{\omega}^2}{k^2}\right)}{(1 + \lambda\bar{\omega} - \theta)} \right] - \frac{2k^2}{r^2} \left[- \frac{\left(1 + \lambda\bar{\omega} + \frac{\alpha\bar{\omega}^2}{2k^2}\right)}{(1 + \lambda\bar{\omega} - \theta)} \right] - \frac{3}{r^4}$$

such that $(1 + \lambda\bar{\omega} - \theta) \neq 0$.

Again, as in Chapter 2,

$$\theta = \frac{\sigma}{2Q}, \quad \lambda = \frac{\eta}{Q} \text{ and } \alpha = \frac{\rho}{Q}.$$

Let, $$b = - \frac{\left(1 + \lambda\bar{\omega} + \frac{\alpha\bar{\omega}^2}{2k^2}\right)}{(1 + \lambda\bar{\omega} - \theta)}$$

$$c = \frac{\left(1 + \lambda\bar{\omega} + \theta + \frac{\alpha\bar{\omega}^2}{k^2}\right)}{(1 + \lambda\bar{\omega} - \theta)}.$$

Observing that $(1 + c) = -2b$, the preceeding equation (3.13) may be written as

$$\left[f_1'''' + \frac{2}{r} f_1''' - \frac{3}{r^2} f_1'' + \frac{3}{r^3} f_1' - \frac{3}{r^4} f_1 \right]$$

$$+ 2k^2 b \left[f_1'' + \frac{1}{r} f_1' - \frac{1}{r^2} \right] + k^4 c f_1 = 0 \qquad \ldots (3.14)$$

Introducing the operator

$$L^2 = \frac{d}{dr} \left[\frac{1}{r} \cdot \frac{d}{dr} (r \ldots) \right] = \left(\frac{d^2}{dr^2} + \frac{1}{r} \cdot \frac{d}{dr} - \frac{1}{r^2} \right)$$

the equation (3.14) simplifies to

$$(L^4 + 2k^2 b L^2 + k^4 c) f_1 = 0$$

or $$(L^2 + k^2 \beta_1^2)(L^2 + k^2 \beta_2^2) f_1 = 0 \qquad \ldots (3.15)$$

in which β_1 and β_2 can be represented as

$$\beta_1 , \beta_2 = \left[b \pm \sqrt{(b^2 - c)} \right]^{1/2}$$

Noting the relation $2b = -(1 + c)$, it is evident that

$$\beta_1 = \sqrt{-c} , \qquad \beta_2 = i \qquad \ldots (3.16)$$

The assumed velocity field (3.11) identically satisfies boundary conditions (3.2a). With the substitution of this

velocity field and the use of the constitutive relation (3.9), the boundary conditions (3.3a) become

$$\left.\begin{aligned} (2Q + 2\eta\bar{\omega})\, f_1' + f_3 &= 0 \\ \text{and} \quad \left(Q + \eta\bar{\omega} - \frac{\sigma}{2}\right)(-kf_1 + f_2') &= 0 \end{aligned}\right\} r = a \quad \ldots (3.17)$$

which, after making use of the equations (3.12b and 3.12c) assume the following form

$$\left.\begin{aligned} \frac{d}{dr}(r\, L^2 f_1) + (2b - 1)\, k^2 r\, \frac{df_1}{dr} - (1+e)\, k^2 f_1 &= 0 \\ (L^2 + k^2)\, f_1 &= 0 \end{aligned}\right\} r = a$$

$$\ldots (3.18)$$

in which,

$$e = \frac{\dfrac{\alpha\,\bar{\omega}^2}{k^2}}{(1 + \lambda\bar{\omega} - \theta)}$$

and, b and c have already been defined.

Two distinct cases may arise in the solution of the equation (3.15) :

(i) $\beta_1 \neq \beta_2$ and (ii) $\beta_1 = \beta_2 = i$.

Case I : $\beta_1 \neq \beta_2$

The general solution of the equation (3.15) may be written in the following form

$$f_1 = A_1 J_1 (k \beta_1 r) + A_2 J_1 (k \beta_2 r) + A_3 Y_1 (k \beta_1 r) + A_4 Y_1 (k \beta_2 r) \qquad \ldots (3.19)$$

where, A_i ($i = 1,2,3,4$) are the constants of integration and, J_1 and Y_1 are the Bessel functions of the first and second kind , respectively. Since $f_1(r)$ should be finite at $r = 0$, A_3 and A_4 must vanish, and equation (3.19) reduces to

$$f_1 = A_1 J_1 (k \beta_1 r) + A_2 J_1 (k \beta_2 r) \qquad \ldots (3.20)$$

The constants A_1 and A_2 are to be determined from the boundary conditions (3.18). Thus, by substitution, the following two equations are obtained:

$$\left.\begin{aligned} & [-(2b+e) J_1 (k \beta_1 a) - k \beta_1 a (1-2b+\beta_1^2) J_o (k \beta_1 a)] A_1 \\ & + [-(2b+e) J_1 (k \beta_2 a) - k \beta_2 a (1-2b+\beta_2^2) J_o (k \beta_2 a)] A_2 \\ & \qquad = 0 \\ & [(1-\beta_1^2) J_1 (k \beta_1 a)] A_1 + [(1-\beta_2^2) J_1 (k \beta_2 a)] A_2 = 0 \end{aligned}\right\} \qquad \ldots (3.21)$$

For instability, there must exist a non-trivial solution of (3.21) , i.e. the characteristic determinant of the co-efficients A_1 and A_2 must be equal to zero. Since $\beta_2 = i$ and $\beta_1 = \sqrt{-c}$ and, in general β_1 should be taken as a complex quantity, the characteristic equation is

$$\frac{(1+c)\, J_1(k\beta_1 a)}{2J_1(k\beta_2 a)} = \frac{(2b+e)\, J_1(k\beta_1 a) + 2k\beta_1 a\, J_o(k\beta_1 a)}{(2b+e)\, J_1(k\beta_2 a) + (1+c)\, k\beta_2 a\, J_o(k\beta_2 a)} \qquad \ldots (3.22)$$

in which, b, c and e have already been defined earlier.

<u>Case II : $\beta_1 = \beta_2 = i$:</u>

The general solution of the equation (3.15), non-singular at $r = 0$, may be written as,

$$f_1 = A_1 I_1(kr) + A_2 r\, I_o(kr) \qquad \ldots (3.23)$$

where, I_1 and I_o are modified Bessel Functions of first kind.

A similar procedure as in the Case I, leads to the following condition for instability:

$$[\,1 + (kr)^2]\; I_1^2(kr) = (kr)^2\; I_o^2(kr) \qquad \ldots (3.24)$$

The above condition is satisfied for $kr = 0$ only (i.e. $r/L = 0$). That it is so can also be seen graphically as in Fig. 3.1. Hence, the loss of stability is precluded whenever $\beta_1 = \beta_2$.

3.3 NUMERICAL COMPUTATIONS

As in Chapter 2, in addition to θ (the non-dimensional stress parameter $\frac{\sigma}{2Q}$), two additional non-dimensional parameters are introduced to facilitate numerical computations. These are

$$\Omega = \left[\frac{\rho \; L^2}{Q} \right]^{1/2} \omega \; ; \quad \lambda_1 = \left[\frac{1}{\rho \; Q \; L^2} \right]^{1/2} \eta \qquad \ldots (3.25)$$

These two parameters are referred to as non-dimensional frequency and non-dimensional viscosity in the subsequent discussions. Hence, β_1 in equation (3.16) will read as

$$\beta_1 = \sqrt{-c} = \sqrt{\left[- \frac{1 + i \lambda_1 \Omega + \theta - \dfrac{\Omega^2}{n^2\pi^2}}{1 + i \lambda_1 \Omega - \theta} \right]} \qquad \ldots (3.26)$$

The equation (3.22) expresses the relation between the non-dimensional stress θ and the frequency parameter Ω which , in general, is a complex quantity. It should be noted that , for a particular value of the stress, if the imaginary part of the frequency Ω is positive, the motion is stable and, if the imaginary part is negative, the motion of the system becomes unstable at that stress ; this is regardless of the sign and the magnitude of the real part of the parameter Ω. Hence, it may be stated that the value of stress will be termed critical below and above which the imaginary part of the frequency parameter is positive and

negative, respectively. It has been observed that, during this transition, the real part of the frequency is very small. Such a calculation is shown in Table 3.1 for $a/L = 1.0$; $\lambda_1 = 0.5$. For such a case, the critical value of θ comes out to be -2.919896.

TABLE 3.1 : VARIATION OF FREQUENCY WITH STRESS NEAR THE CRITICAL STRESS FOR $\lambda_1=0.5$; $\theta_{cr}= -2.919896$

Stress Parameter θ	Frequency Parameter Ω	Remarks
-2.819696	1.21×10^{-9}, 6.92×10^{-2}	Below critical
-2.919696	3.35×10^{-10}, 6.59×10^{-2}	Below critical
-2.919896	1.68×10^{-15}, 2.84×10^{-7}	Critical
-2.919706	1.78×10^{-15}, -7.13×10^{-6}	Above critical
-3.019696	1.63×10^{-10}, -6.78×10^{-2}	Above critical

Such critical values of θ, the non-dimensional stress $\frac{\sigma}{2Q}$, obtained for different values of $\frac{a}{L}$, are tabulated in Table 3.2 and represented graphically in Fig. 3.2. It may be seen that as $\frac{a}{L} \to 0$, the value of the compressive stress tends to a large value ; it is obviously so because the motion is constrained to be in the

axisymmetric mode. Actually for smaller value of $\frac{a}{L}$, the instability is likely to be in the asymmetric mode.

TABLE 3.2 : VARIATION OF STRESS PARAMETER $\theta = \frac{\sigma}{2Q}$ WITH $\frac{a}{L}$

$\frac{a}{L}$	β_1	Stress parameter θ
0.8	0.94971, 0	- 18.4548
1.0	0.69986, 0	- 2.9199
1.2	0.52441, 0	- 1.7586
1.4	0.39247, 0	- 1.3642
1.6	0.28967, 0	- 1.1776
1.8	0.18910, 0	- 1.0742
2.0	0.07437, 0	- 1.0111
2.2	0 , 0.12304	- 0.9701
2.4	0 , 0.15208	- 0.9548

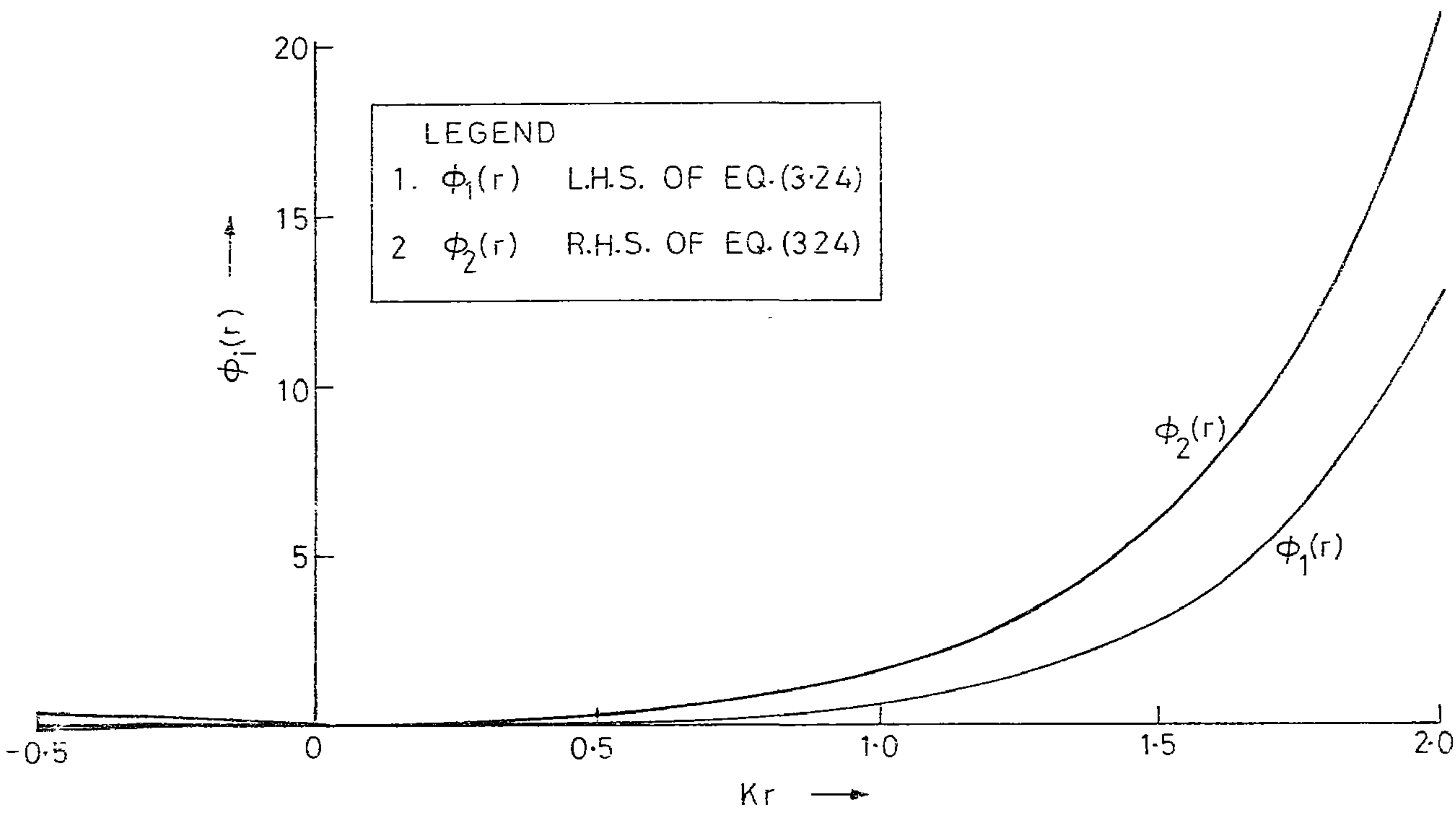


Fig.3·1 Plot of $\phi_i(r)$ vs. Kr

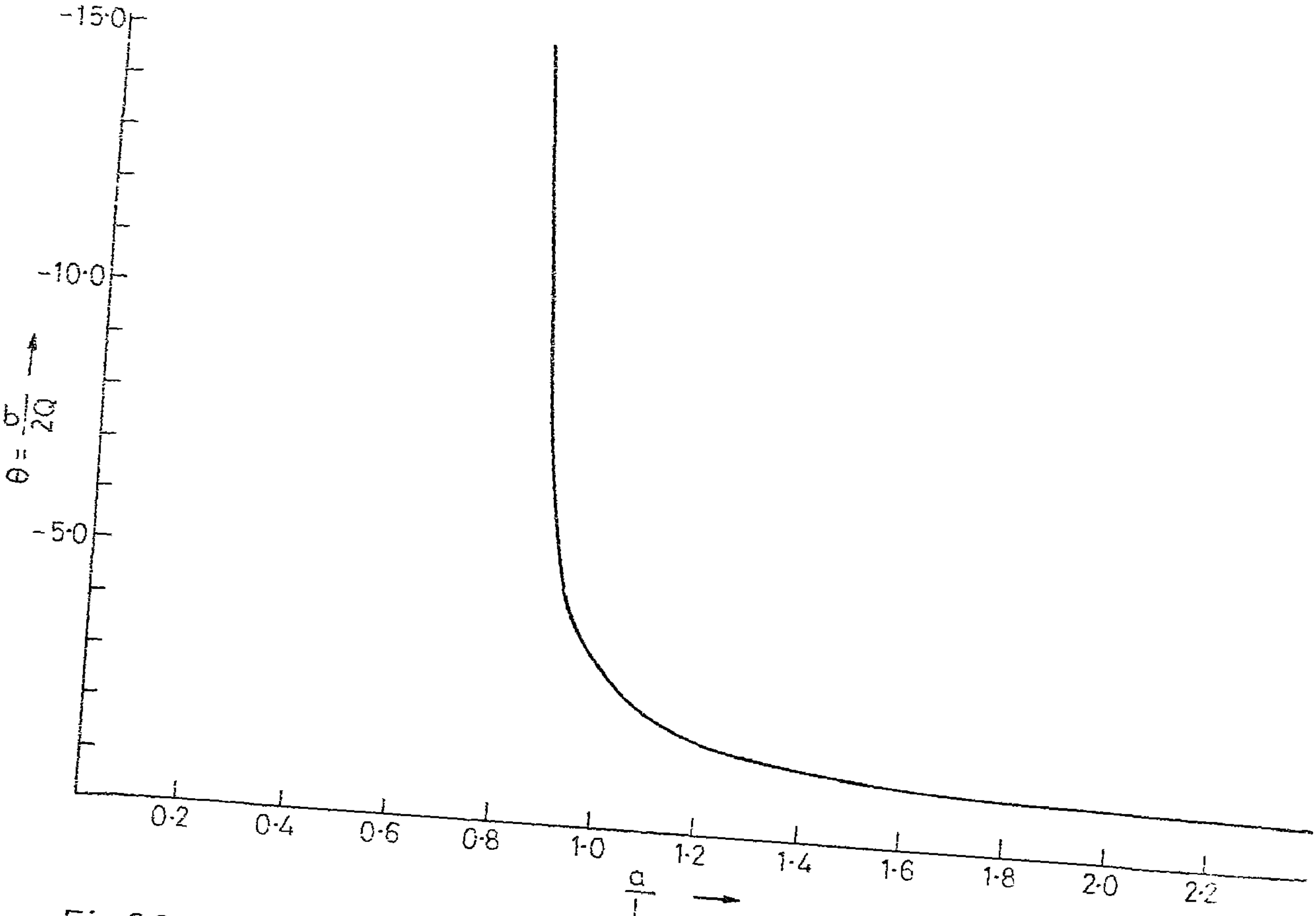


Fig.3·2 Variation of critical stress in compression with a/L

CHAPTER : 4

CHAPTER : 4

STABILITY OF A THIN-WALLED VISCOELASTIC CYLINDER UNDER TORSION

4.1 INTRODUCTION AND PROBLEM FORMULATION

In this chapter, the critical state of stress for instability in a geometrically perfect viscoelastic thin cylindrical shell under torsion is established. Although, the problem of stability of geometrically perfect elastic and elastic-plastic thin cylindrical shells received the attentions of many researchers, it appears that the corresponding problem for a viscoelastic material has not yet been investigated. Here, the basic formulation developed by Neale (1973) for elastic-plastic cylinders has been extended to include the case of viscoelastic cylinders.

A geometrically perfect thin viscoelastic cylindrical shell is subjected to a torque M applied at both ends. At any instant of time, let R , t and L be the mean radius, thickness and length of the shell, such that $\frac{t}{R} << 1$ and $\frac{R}{L} << 1$. If the boundary conditions are such that the cylindrical form of the shell is preserved during its motion, the only nonvanishing stress components, with reference to a cylindrical polar coordinate (r , θ, z) are

$$\sigma_{\theta z} = \sigma_{z\theta} = \frac{M}{2\pi R^2 t} = \tau \qquad \dots (4.1)$$

all other $\sigma_{ij} = 0$.

In the current state of equilibrium, it is assumed that the cylindrical surfaces are free from any nominal traction-rate , i.e. ,

$$\dot{s}_{rr} = \dot{s}_{r\theta} = \dot{s}_{rz} = 0 \quad \text{at} \quad r = R - \frac{t}{2} \text{ and at } r = R + \frac{t}{2} \qquad \ldots (4.2)$$

Using the relation (1.39) between $\dot{s}_{ij}$ and $\frac{D\tau_{ij}}{Dt}$, conditions (4.2) imply that

$$\left.\begin{aligned} &\frac{D\tau_{rr}}{Dt} = 0 \\ &\frac{D\tau_{r\theta}}{Dt} - \sigma_{\theta z} \cdot \epsilon_{rz} = 0 \\ &\frac{D\tau_{rz}}{Dt} - \sigma_{\theta z} \cdot \epsilon_{r\theta} = 0 \end{aligned}\right\} \qquad \ldots (4.3)$$

In view of the small thickness of the cylinder, the preceding relations may be assumed to hold for all values of τ ; consequently

$$\frac{D\tau_{rr}}{Dt} = \frac{D\tau_{r\theta}}{Dt} = \frac{D\tau_{rz}}{Dt} = 0 \text{ , at all } r \qquad \ldots (4.4)$$

which suggests that $\epsilon_{rz} = \epsilon_{r\theta} = 0$, as is the case in thin shell analysis.

The components of the strain-rate tensor

$e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ and those of the rotation tensor

$\omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ in polar coordinate are given by

$$e_{ij} = \begin{bmatrix} \frac{\partial u}{\partial r} & \frac{1}{2}\left(\frac{1}{r} \cdot \frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \\ \frac{1}{2}\left(\frac{1}{r} \cdot \frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right) & \left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \quad \ldots (4.5)$$

$$\omega_{ij} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial r} + \frac{v}{r}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) \\ -\frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial r} + \frac{v}{r}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{1}{r}\frac{\partial w}{\partial \theta}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right) & -\frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{1}{r}\frac{\partial w}{\partial \theta}\right) & 0 \end{bmatrix} \quad \ldots (4.6)$$

In view of the stress distribution (4.1) and using the expressions (4.5) for e_{ij}, the governing constitutive equation (1.40) can be expressed in the following form in terms of the physical velocity components u, v and w:

$$
\left.
\begin{aligned}
\dot{s}_{rr} &= (2Q + 2\eta D)\, u_{,r} + p \\
\dot{s}_{\theta\theta} &= (2Q + 2\eta D)\left(\frac{u}{r} + \frac{1}{r} v_{,\theta}\right) + p - \tau \frac{1}{r} w_{,\theta} \\
\dot{s}_{zz} &= (2Q + 2\eta D)\, w_{,z} + p - \tau\, v_{,z} \\[1ex]
\dot{s}_{r\theta} &= (Q + \eta D)\left(\frac{1}{r} u_{,\theta} - \frac{v}{r} + v_{,r}\right) - \tau \frac{1}{2}(u_{,z} + w_{,r}) \\
\dot{s}_{\theta r} &= (Q + \eta D)\left(\frac{1}{r} u_{,\theta} - \frac{v}{r} + v_{,r}\right) + \tau \frac{1}{2}(u_{,z} - w_{,r}) \\
\dot{s}_{rz} &= (Q + \eta D)(u_{,z} + w_{,r}) - \tau \frac{1}{2}\left(\frac{1}{r} u_{,\theta} - \frac{v}{r} + v_{,r}\right) \\
\dot{s}_{zr} &= (Q + \eta D)(u_{,z} + w_{,r}) + \tau \frac{1}{2}\left(\frac{1}{r} u_{,\theta} + \frac{v}{r} - v_{,r}\right) \\[1ex]
\dot{s}_{\theta z} &= (Q + \eta D)\left(v_{,z} + \frac{1}{r} w_{,\theta}\right) - \tau\left(\frac{u}{r} + \frac{1}{r} v_{,\theta}\right) \\
\dot{s}_{z\theta} &= (Q + \eta D)\left(v_{,z} + \frac{1}{r} w_{,\theta}\right) - \tau\, w_{,z}
\end{aligned}
\right\}
\qquad \ldots (4.7)
$$

where, as before, a comma signifies partial differentiation with respect to the suffix that follows and $D \equiv \frac{\partial}{\partial t}$.

4.2 STABILITY CONDITION

A general stability criterion and its derivation are presented in Chapter 1. For the sake of convenient reference, the criterion (1.43) can be expressed as a requirement that the functional

$$I = \int_V \left[\frac{1}{2} \rho \dot{v}_j^2 + \frac{1}{2} \dot{s}_{ij} v_{j,i} \right] dV \qquad \ldots (4.8)$$

be positive for all continuous velocity fields v_i. Substitution of the rate constitutive equations (4.7) in the functional (4.8) leads to

$$\iiint \frac{1}{2} \rho \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right] r\, dr\, d\theta\, dz$$

$$+ \iiint \Big[(\bar{Q} + \eta D) \{ 2u_{,r}^2 + 2\left(\frac{u}{r} + \frac{1}{r} v_{,\theta} \right)^2 + 2w_{,z}^2$$

$$+ \left(\frac{1}{r} u_{,\theta} - \frac{v}{r} + v_{,r} \right)^2 + \left(v_{,z} + \frac{1}{r} w_{,\theta} \right)^2 + (u_{,z} + w_{,r})^2 \}$$

$$- \bar{\tau} \{ \frac{2}{r} w_{,\theta} \left(\frac{u}{r} + \frac{1}{r} v_{,\theta} \right) + 2v_{,z} \cdot w_{,z}$$

$$- \frac{3}{2r} u_{,\theta} (u_{,z} - w_{,r}) + \frac{1}{2} \left(v_{,r} - \frac{v}{r} \right) w_{,r} \} \Big] r\, dr\, d\theta\, dz$$

$$\ldots (4.9)$$

Taking into consideration the usual assumptions of the thin shell theory (cf., e.g. Novozhilov 1953) the velocity fields are expressed as

$$\left.\begin{aligned} u &= \bar{u} \\ v &= \bar{v} + \frac{\xi}{R}\left(\bar{v} - u_{,\theta}\right) \\ &= \bar{v} + \frac{\xi}{R}\left(\bar{v} - \bar{u}_{,\theta}\right) \\ w &= \bar{w} - \xi\, u_{,z} \\ &= \bar{w} - \xi\, \bar{u}_{,z} \end{aligned}\right\} \qquad \ldots (4.10)$$

where $\bar{u}(\theta,z)$, $\bar{v}(\theta,z)$, $\bar{w}(\theta,z)$ are the velocity components at the middle surface of the cylinder along the r -, θ -, z - axes respectively, and $\xi = r - R$, represents the distance from the middle surface. A substitution of the velocity fields (4.10) into (4.9) and subsequent integration through the thickness of the shell leads to the following condition for stability:

$$\iint R\, t\, \rho\, \Big[\, \dot{\bar{u}}^2 + \dot{\bar{u}}_{,\theta}^2 + 2(1+\bar{t})\, \dot{\bar{u}}_{,\theta}\left(\dot{\bar{v}} - \dot{\bar{u}}_{,\theta}\right) + (1+3\bar{t})\left(\dot{\bar{v}} - \dot{\bar{u}}_{,\theta}\right)^2$$

$$+ \left(\dot{\bar{w}} + R\,\dot{\bar{u}}_{,z}\right)^2 - 2t(1+\bar{t})\,\dot{\bar{u}}_{,z}\left(\dot{\bar{w}} + R\,\dot{\bar{u}}_{,z}\right)$$

$$+ R^2\,(1+3\bar{t})\,\dot{\bar{u}}_{,z}^2\,\Big]\, d\theta\, dz$$

Eq. 4.11 contd..

$$+ \iint (Q + \eta D) \Big[2 (\bar{u} + \bar{u}_{,\theta\theta})^2 \ln \left(\frac{1 + t/2R}{1 - t/2R}\right)$$

$$+ 2(\bar{v}_{,\theta} - \bar{u}_{,\theta\theta})^2 \left(\frac{t}{R}\right)$$

$$+ 4(\bar{u} + \bar{u}_{,\theta\theta}) (\bar{v}_{,\theta} - \bar{u}_{,\theta\theta}) \left(\frac{t}{R}\right)$$

$$+ 2Rt(\bar{w}_{,z} + R\,\bar{u}_{,zz})^2 + 2R^3t\,(1+3\bar{t})\,\bar{u}_{,zz}$$

$$- 4R^2t(1+\bar{t})\,(\bar{u}_{,zz})\,(\bar{w}_{,z} + R\,\bar{u}_{,zz})$$

$$+ (\bar{v} - \bar{u}_{,\theta})^2 \left(\frac{t}{R}\right)$$

$$+ 4Rt\,(\bar{v}_{,z} - \bar{u}_{,\theta z}) + (\bar{w}_{,\theta} + R\,\bar{u}_{,\theta z})^2 \ln \left(\frac{1+t/2R}{1-t/2R}\right)$$

$$+ 2t\,(\bar{v}_{,z} - \bar{u}_{,\theta z})(\bar{w}_{,\theta} + R\,\bar{u}_{,\theta z})$$

$$+ Rt\,\bar{u}^2_{,z}\Big]\, d\theta\, dz$$

$$- \iint \tau \Big[2t \Big\{\frac{1}{R}\,\bar{w}_{,\theta}\,(\bar{v}_{,\theta} - \bar{u}_{,\theta\theta}) - (\bar{u} + \bar{u}_{,\theta\theta})\,\bar{u}_{,\theta z}\Big\}$$

$$+ 2 \ln \left(\frac{1 + t/2R}{1 - t/2R}\right) (\bar{u} + \bar{u}_{,\theta\theta})\,(\bar{w}_{,\theta} + R\,\bar{u}_{,\theta z})$$

$$+ 2R^2t\,\Big\{\bar{u}_{,\theta z}\left(\frac{1}{R}\bar{w}_{,z} + \bar{u}_{,zz}\right)$$

$$+ (1+\bar{t})\,(\bar{v}_{,z} - \bar{u}_{,\theta z})\left(\frac{1}{R}\bar{w}_{,z} + \bar{u}_{,zz}\right)$$

$$- (1+\bar{t})\,\bar{u}_{,\theta z}\,\bar{u}_{,zz} - \bar{u}_{,zz}\,(\bar{v}_{,z} - \bar{u}_{,\theta z})\Big\}$$

$$- \frac{3}{2}\,t\,\bar{u}_{,\theta}\,\bar{u}_{,z}\Big]\, d\theta\, dz > 0 \qquad \ldots (4.11)$$

in which $\bar{t} = \dfrac{t^2}{12R^2}$.

The critical velocity distributions $\bar{u}, \bar{v}, \bar{w}$ are those which minimize the forgoing functional and, therefore, the Euler equations of these integral must be satisfied. Under torsional loading, the deformation pattern of the viscoelastic cylindrical shell may be assumed to consist of a number of circumferential wave like patterns which spiral around the cylinder from one end to other ; a solution may be sought in the following form

$$\left.\begin{aligned} \bar{u} &= A_1 \sin\left(\frac{\Psi z}{R} - n\theta\right) \exp(\bar{\omega} t) \\ \bar{v} &= A_2 \cos\left(\frac{\Psi z}{R} - n\theta\right) \exp(\bar{\omega} t) \\ \bar{w} &= A_3 \cos\left(\frac{\Psi z}{R} - n\theta\right) \exp(\bar{\omega} t) \end{aligned}\right\} \quad \ldots (4.12)$$

where $\Psi = \dfrac{m\pi R}{L}$, and m, n are positive integers

$\bar{\omega} = i\omega$, $i = \sqrt{-1}$, ω being the frequency of motion.

Substitution of (4.12) in (4.11) results in,

$$\left(\frac{\pi L t}{R}\right)(\rho\, \bar{\omega}^2 R^2)\,[\,A_1^2 + n^2 A_1^2 + n\,A_1 A_2 + A_2^2$$
$$+ (\Psi A_1 + A_2)^2 - 2(1 + \bar{t})\,\Psi A_1\,(\Psi A_1 + A_3)$$
$$+ (1 + 3\bar{t})\,\Psi^2 A_1^2\,]$$

Eq. 4.13 contd..

$$+ \left(\frac{\pi L t}{R}\right) (Q + \eta \bar{\omega}) \Big[\bar{T} \; \{2(1-n^2)^2 A_1^2\} + \{2n^2 (nA_1 + A_2)^2\}$$

$$+ \{4n(1-n^2)(n A_1 + A_2) A_1\}$$

$$+ \{2\Psi^2 (\Psi A_1 + A_3)^2 + 2(1+3\bar{t}) \Psi^4 A_1^2$$

$$- 4(1+\bar{t}) \Psi^3 A_1 (\Psi A_1 + A_3)\}$$

$$+ \{(n A_1 + A_2)^2\}$$

$$+ \{4(n A_1 - A_2)^2 + \bar{T} (n \Psi A_1 + n A_3)^2$$

$$- 2n \Psi (n A_1 + A_2)(A_1 + A_3)\}$$

$$+ \Psi^2 A_1^2 \Big]$$

$$- \left(\frac{\pi L t}{R}\right) \tau \Big[2 \{n^2 A_3 (n A_1 + A_2) - n\Psi (1-n^2) A_1^2\}$$

$$+ \bar{T} \{2n (1-n^2) A_1 (A_3 + \Psi A_1)\}$$

$$- 2n \Psi^2 A_1 (\Psi A_1 + A_3)$$

$$+ 2(1 + \bar{t}) \{\Psi^2 (n A_1 + A_2)(\Psi A_1 + A_3) + n \Psi^3 A_1^2\}$$

$$- 2\Psi^3 A_1 (n A_1 + A_2) + \frac{3}{2} n \Psi A_1^2 \Big]$$

$$> \quad 0 \qquad \ldots (4.13)$$

in which

$$\bar{T} = \frac{\ln\left[\dfrac{1 + t/2R}{1 - t/2R}\right]}{\dfrac{t}{R}}$$

The Euler equations of (4.13) yield three linear homogeneous equations in A_1, A_2 and A_3, which on simplification, leads to

$$\left.\begin{aligned} P_{11}\,A_1 + P_{12}\,A_2 + P_{13}\,A_3 &= 0 \\ P_{21}\,A_1 + P_{22}\,A_2 + P_{23}\,A_3 &= 0 \\ P_{31}\,A_1 + P_{32}\,A_2 + P_{33}\,A_3 &= 0 \end{aligned}\right\} \quad \ldots (4.14)$$

where,

$$\begin{aligned} P_{11} = {} & \left(1 + \frac{\eta\,\bar{\omega}}{Q}\right) \left[\{4(1-n^2)^2 + 2n^2\,\Psi^2\}\,\bar{T} + 4\Psi^4\bar{t} \right. \\ & \left. + 22n^2 - 4n^4 - 4n^2\,\Psi + 2\Psi^2 \right] \\ & - \left(\frac{\tau}{Q}\right) \left[4n\,\Psi\,(1-n^2)\,\bar{T} + 8\,n\,\Psi^3\,\bar{t} + 4n^3\,\Psi - n\,\Psi \right] \\ & + \left(\frac{\rho\,\bar{\omega}^2 R^2}{Q}\right) \left[2(1+n^2) + 2\bar{t}\,\Psi^2 \right] \end{aligned}$$

$$P_{12} = \left(1 + \frac{\eta\,\bar{\omega}}{Q}\right)(-2n - 2n\,\Psi) - \left(\frac{\tau}{Q}\right)(2\Psi^3\,\bar{t}) + \left(\frac{\rho\bar{\omega}^2 R^2}{Q}\right)(n)$$

$$\begin{aligned} P_{13} = {} & \left(1 + \frac{\eta\,\bar{\omega}}{Q}\right)(2n^2\,\Psi\,\bar{T} - 4\,\Psi^3\,\bar{t} - 2n^2\Psi) \\ & - \left(\frac{\tau}{Q}\right) \left[2n^3 + 2n\,(1-n^2)\,\bar{T} + 2n\Psi^2\,\bar{t} \right] \\ & + \left(\frac{\rho\,\bar{\omega}^2 R^2}{Q}\right) \left[-2\,(1 + \bar{t}) \right] \end{aligned}$$

$$P_{21} = (1 + \frac{\eta\,\bar{\omega}}{Q})\ (-2n - 2n\,\Psi + 2\Psi^2) - (\frac{\tau}{Q})\ (2\Psi^3\,\bar{t}\,)$$
$$+ (\frac{\rho\,\bar{\omega}^2 R^2}{Q})\ (n)$$

$$P_{22} = (1 + \frac{\eta\,\bar{\omega}}{Q})\ (4n^2 + 10) + (\frac{\rho\,\bar{\omega}^2 R^2}{Q})\ (2)$$

$$P_{23} = (1 + \frac{\eta\,\bar{\omega}}{Q})\ (-2\,n\Psi) - (\frac{\tau}{Q})\ [\,2n^2 + 2\Psi^2\,(1 + \bar{t}\,)]$$

$$P_{31} = (1 + \frac{\eta\,\bar{\omega}}{Q})\ (2n^2\,\Psi\,\bar{T} - 4\Psi^3\,\bar{t} - 2n^2\,\Psi)$$
$$- (\frac{\tau}{Q})\ [\,2n^3 + 2n\,(1-n^2)\,\bar{T} + 2n\,\Psi^2\,\bar{t}\,]$$
$$+ (\frac{\rho\,\bar{\omega}^2 R^2}{Q})\ (-2\Psi\,\bar{t}\,)$$

$$P_{32} = (1 + \frac{\eta\,\bar{\omega}}{Q})\ (-2n\,\Psi\,) - (\frac{\tau}{Q})\ [\,2n^2 + 2\Psi^2\,(\,1 + \bar{t}\,)]$$

$$P_{33} = (1 + \frac{\eta\,\bar{\omega}}{Q})(4\,\Psi^2 + 2n\,\bar{T}\,) + (\frac{\rho\,\bar{\omega}^2 R^2}{Q})\ (2)$$

For the existence of a non-trivial solution of (4.14), the determinant of the co-efficients A_i of the matrix must vanish, i.e.,

$$\begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{vmatrix} = 0 \qquad \ldots (4.15)$$

This leads to a cubic equation in non-dimensional stress τ/Q, viscosity and frequency (after non-dimensionalization), which has to be solved for a particular set of geometric and material parameters. The behaviour of the load-frequency relationship has to be studied by varying the stress τ/Q gradually, keeping the other parameters fixed. If it is found that the imaginary part of the frequency is positive, for a particular value of stress, the deformation is stable and when the imaginary part becomes negative the deformation is in an unstable range. The stress at which such a transition takes place is termed as the critical stress.

4.3 NUMERICAL RESULTS AND DISCUSSIONS

For numerical computations, the non-dimensionalised stress, frequency and viscosity parameters are introduced as follows:

$$\theta = \frac{\tau}{2Q} \quad ; \Omega = \sqrt{\left(\frac{\rho R^2}{Q}\right)} \quad ; \lambda_1 = \frac{\eta}{\sqrt{(\rho Q R^2)}}$$

As the equation (4.15) is fairly involved, a trial and error scheme has been initially adopted to study the stress-requency response of the system. At a sufficiently, high value of stress it has been observed that the real part of the frequency is zero while its imaginary part is

negative, whereas at a lower value of stress, the real part of the frequency is non-zero and the imaginary part of the frequency is positive. It is therefore, natural to expect that at some value of stress either imaginary part is zero or both the real and the imaginary parts vanish. This value of the stress will be termed critical because it marksthe transition from a stable to an unstable configuration. It was verified at a later stage that there is a value of stress at which the frequency Ω is zero.

Next, for a given R/t ratio, the critical non-dimensionalized stress parameter θ ($\frac{\tau}{2Q}$) has been obtained by minimizing (4.15) with respect to n and Ψ keeping the frequency parameter $\Omega = 0$. The minimizing value of n is always 2 while the minimizing value of Ψ lies between 0.2 - 1.0, higher values corresponding to larger values of R/t.

Once the critical stress parameter is determined for a particular R/t ratio, stress is increased gradually from the value zero to the critical value and slightly beyond and the frequencies are determined for several values of viscosity parameter λ_1 (0.01, 0.1, 0.5) keeping n and Ψ constant/unchanged. These are shown in Tables4.1 and 4.2 for two values of R/t ratio (20, 50). The stress-frequency relationships are also shown schematically in Figs. 4.1 and 4.2. The eleventh row in Tables 4.1 and 4.2 indicates the state of

the frequency parameter when the stress is just critical and the twelfth row shows that the imaginary part of the frequency parameters are negative, when the stress is more than the critical value. Also, the real parts of the frequencies are zero at these stress levels.

The values of the critical shear stress $\frac{\tau}{2\Omega}$ are presented in Table 4.3 and its variation with R/t is shown graphically in Fig. 4.3. The change in critical stress is found to be negligible for higher values of $\frac{R}{t}$ (i.e. $\frac{R}{t} > 100$).

TABLE 4.1 : VARIATION OF FREQUENCY WITH STRESS: $\frac{R}{t} = 20$; $n = 2$; $\Psi = 0.2667$;
$\left(\frac{\tau}{2Q}\right)_{cr} = 5.16 \times 10^{-2}$;

No.	Stress $\frac{\tau}{2Q}$ $\times 10^{-2}$	Frequency Ω_1 with $\lambda_1 = 0.01$ $\times 10^{-1}$	$\times 10^{-4}$	Frequency Ω_2 with $\lambda_1 = 0.10$ $\times 10^{-1}$	$\times 10^{-3}$	Frequency Ω_3 with $\lambda_1 = 0.50$ $\times 10^{-1}$	$\times 10^{-2}$
1	0.000	-2.849	4.057	-2.848	4.057	-2.841	2.029
2	0.516	-2.702	4.057	-2.702	4.057	-2.696	2.029
3	1.032	-2.548	4.057	-2.547	4.057	-2.540	2.029
4	1.548	-2.383	4.057	-2.383	4.057	-2.375	2.029
5	2.064	-2.206	4.057	-2.206	4.057	-2.197	2.029
6	2.580	-2.014	4.057	-2.014	4.057	-2.004	2.029
7	3.096	-1.802	4.057	-1.801	4.057	-1.790	2.029
8	3.612	-1.560	4.057	-1.560	4.057	-1.547	2.029
9	4.128	-1.274	4.057	-1.273	4.057	-1.258	2.029
10	4.664	-0.901	4.057	-0.900	4.057	-0.878	2.029
11	5.160	0.	0.	0.	0.	0.	0.
12	5.676	0.	-896.700	0.	-861.100	0.	-7.205

TABLE 4.2 : VARIATION OF FREQUENCY WITH STRESS : $\frac{R}{t} = 50$; $n = 2$; $\Psi = 0.3752$;

$\left(\frac{\tau}{2Q}\right)_{cr.} = 1.275 \times 10^{-2}$

No.	Stress $\frac{\tau}{2Q}$ $x10^{-2}$	Frequency with $\lambda_1 = 0.01$: Ω_1 $x10^{-1}$	$x10^{-4}$	Frequency with $\lambda_1 = 0.10$: Ω_2 $x10^{-1}$	$x10^{-3}$	Frequency with $\lambda_1 = 0.50$: Ω_3 $x10^{-1}$	$x10^{-2}$
1	0.0000	-3.798	7.213	-3.797	7.213	-3.781	3.606
2	0.1275	-3.603	7.213	-3.602	7.213	-3.585	3.606
3	0.2550	-3.397	7.213	-3.396	7.213	-3.378	3.606
4	0.3825	-3.178	7.213	-3.177	7.213	-3.157	3.606
5	0.5100	-2.942	7.213	-2.941	7.213	-2.920	3.606
6	0.6375	-2.686	7.213	-2.685	7.213	-2.661	3.606
7	0.7650	-2.402	7.213	-2.401	7.213	-2.375	3.606
8	0.8925	-2.080	7.213	-2.079	7.213	-2.049	3.606
9	1.0200	-1.699	7.213	-1.697	7.213	-1.660	3.606
10	1.1475	-1.201	7.213	-1.199	7.213	-1.146	3.606
11	1.2750	0.	0.	0.	0.	0.	0.
12	1.4025	0.	-1194	0.	-131.1	0.	-8.934

TABLE 4.3 : VARIATION OF CRITICAL STRESS WITH $\frac{R}{t}$

$\frac{R}{t}$	n	$\Psi = \frac{m\pi R}{L}$	$\theta_{cr} = \left(\frac{\tau}{2Q}\right)_{cr.}$
10		0.2342	0.1313
20		0.2667	0.0516
30		0.2872	0.0267
40		0.3124	0.0177
50	2	0.3572	0.0128
60		0.4712	0.0098
70		0.5455	0.0096
80		0.6281	0.0094
90		0.7713	0.0093
100		0.8244	0.0091

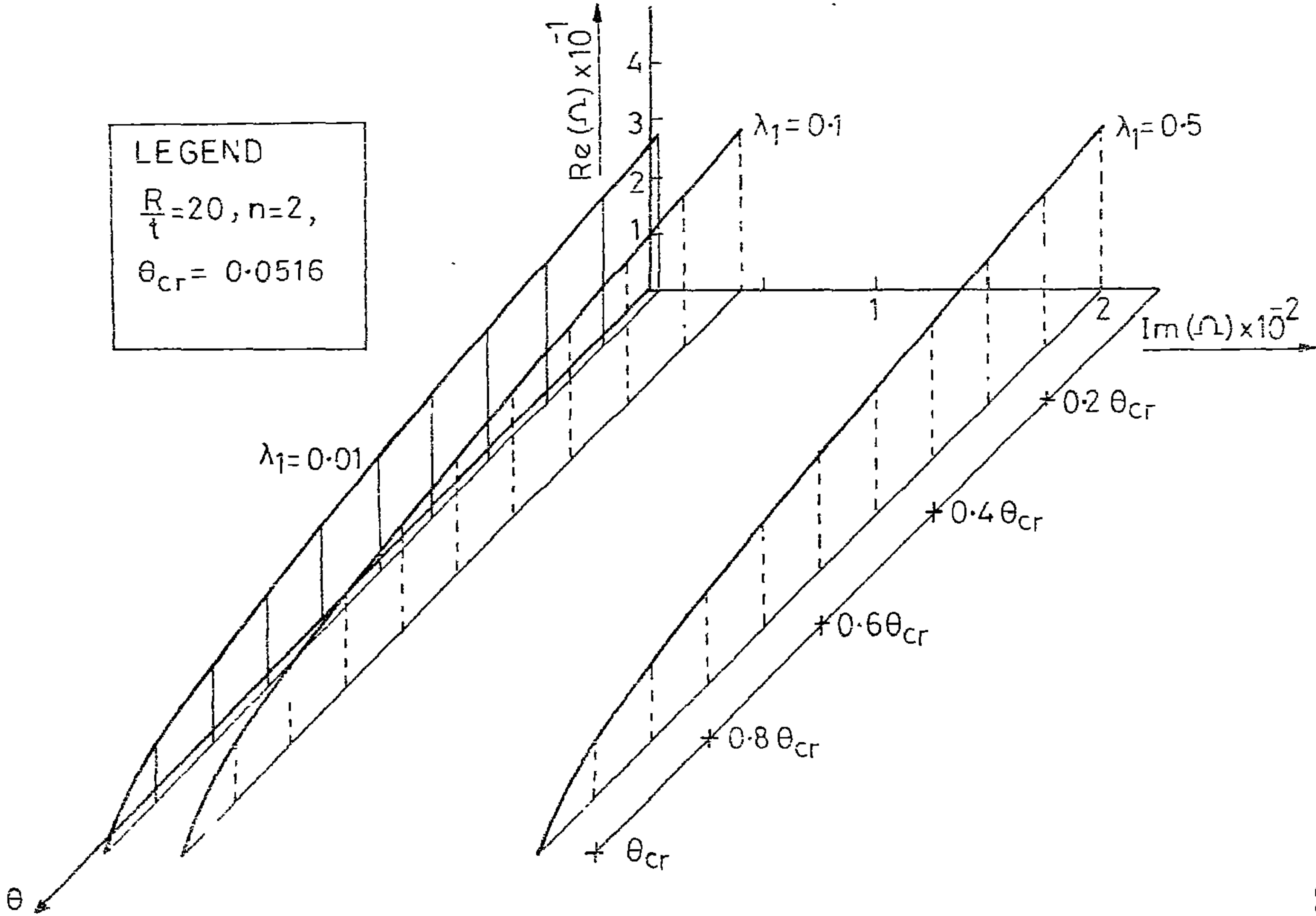


Fig. 4·1 Variation of frequency with shear stress

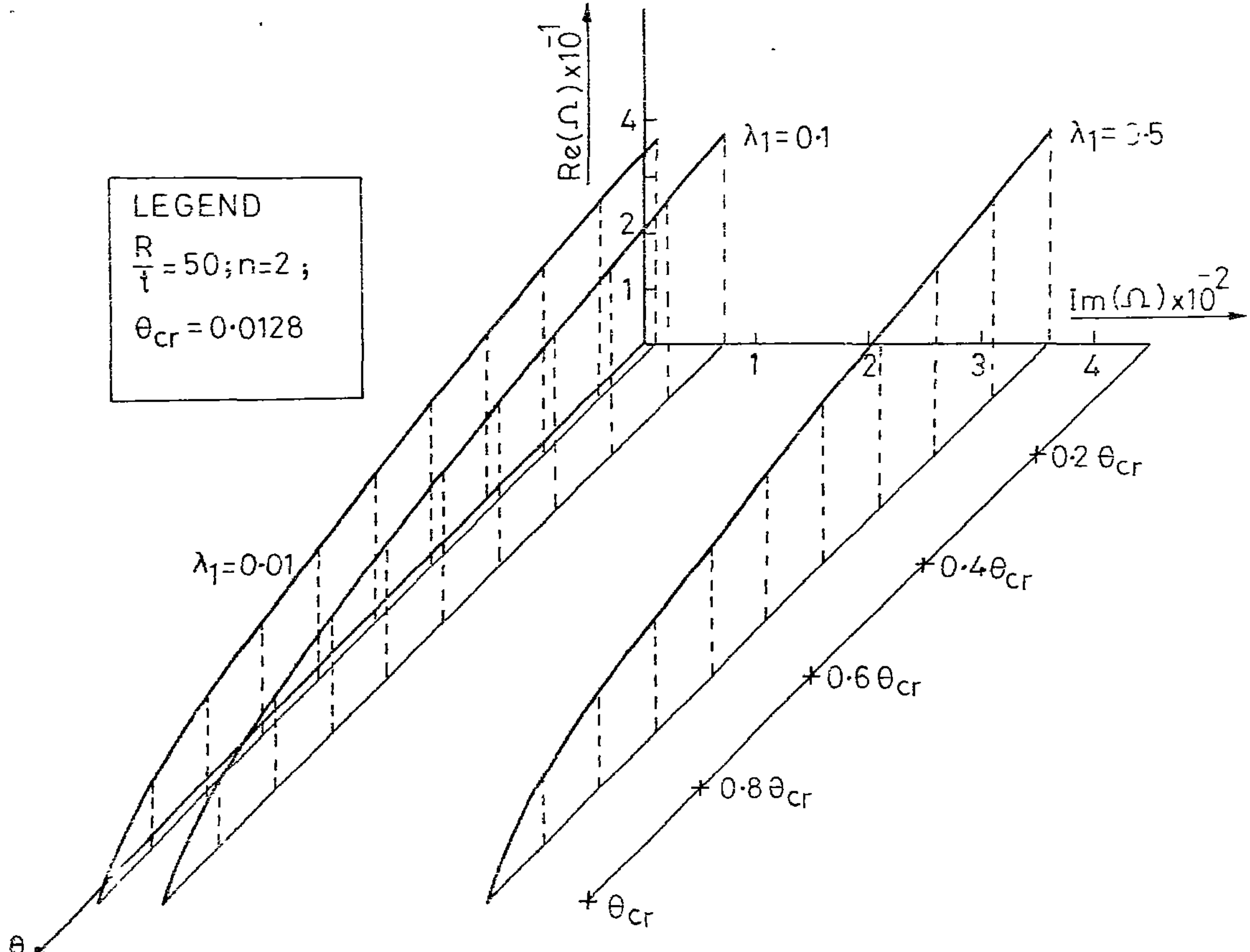


Fig. 4.2 Variation of frequency with shear stress

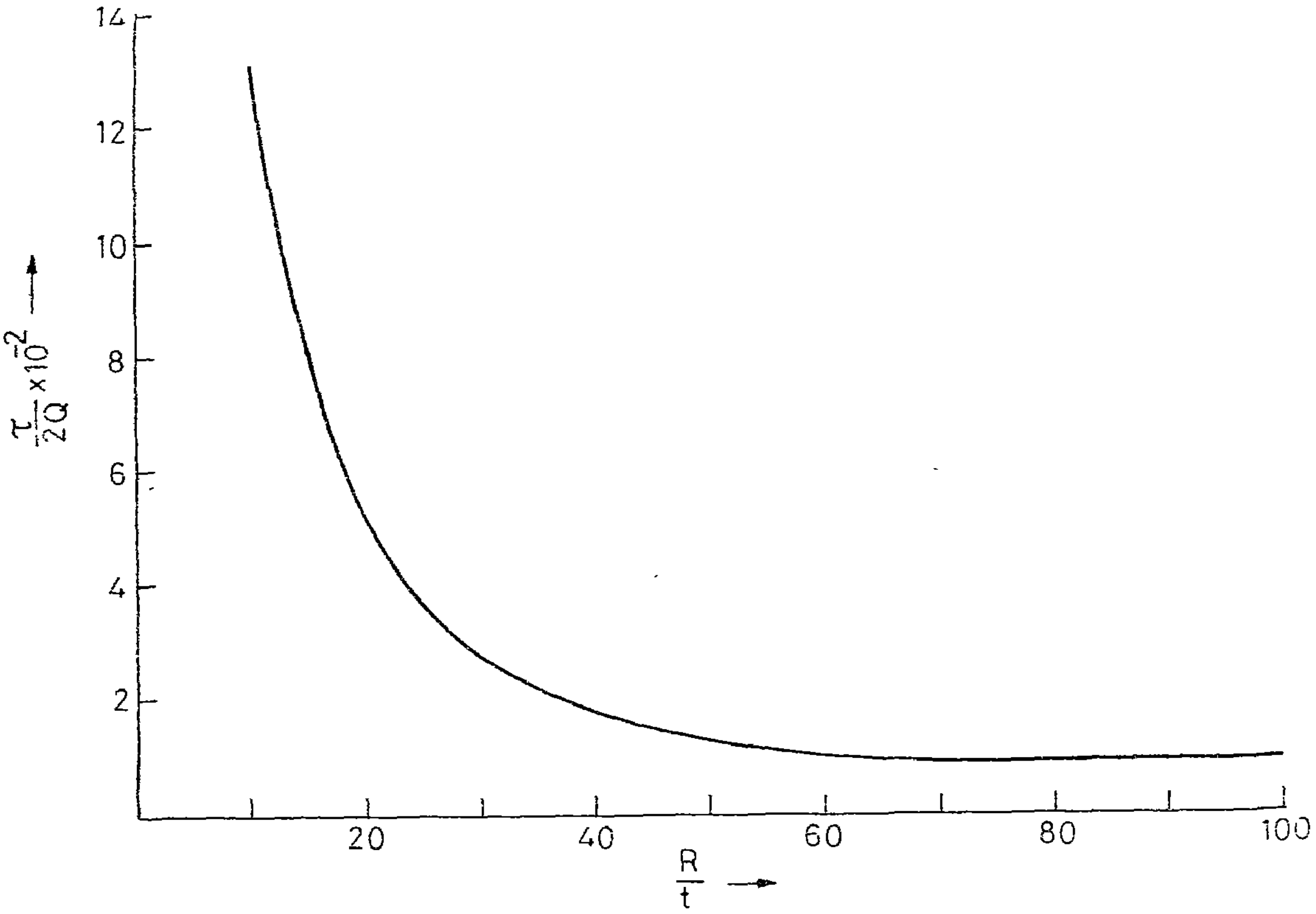


Fig.4·3 Variation of critical shear stress in torsion with R/t

CHAPTER : 5

CHAPTER : 5

STABILITY OF A THIN RECTANGULAR VISCOELASTIC PLATE UNDER INPLANE LOADING

5.1 INTRODUCTION AND PROBLEM FORMULATION

In this chapter, the stability of a thin rectangular viscoelastic plate under inplane direct loading has been investigated. The problem formulation is that of a plane stress condition.

An incompressible homogeneous viscoelastic solid in a state of plane stress is considered during the process of continuing deformation. The current distribution of stress, together with other mechanical properties at this instant is supposed to be already determined. The current shape of the body is rectangular and of dimensions a x b x h. A set of rectangular Cartesian coordinate axes x_i is chosen such that the plate, with its edges $x_1 = 0, a$ and $x_2 = 0, b$ and with the plane faces $x_3 = \pm \frac{h}{2}$, is simply supported on all four edges. Whenever convenient, the coordinate x_i will be replaced by x,y,z and the velocity components v_i by u,v,w, respectively. Referred to this frame, the instanteneous stress field is given by

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \ldots (5.1)$$

During further infinitesimal deformation, when the system is perturbed from the current state to a neighbouring state, it has been assumed that the ends x = 0,a and y = 0,b are frictionless (simply supported). Also the faces $z = \pm \frac{h}{2}$ are assumed to be free of nominal traction rates. These conditions are expressed as

$$\left.\begin{aligned} V_3 &= 0 \\ \dot{T}_j &= n_i \, \dot{s}_{ij} = 0, \; j = 2,3 \end{aligned}\right\} x = 0,a \qquad \text{... (5.2)}$$

$$\left.\begin{aligned} V_3 &= 0 \\ \dot{T}_j &= n_i \, \dot{s}_{ij} = 0 \;, \; j = 1,3 \end{aligned}\right\} y = 0,\, b \qquad \text{... (5.3)}$$

$$\dot{T}_j = n_i \, \dot{s}_{ij} = 0 \qquad z = \pm \frac{h}{2} \qquad \text{... (5.4)}$$

where n_i is the unit outward normal to the boundary surfaces.

For the present case, the rate constitutive equation for the viscoelastic material, as described by (1.40), can be expressed in the following explicit form

$$\dot{s}_{xx} = (2Q + 2\eta D - \sigma_1)\, u_x + p$$

$$\dot{s}_{yy} = (2Q + 2\eta D - \sigma_2)\, v_y + p$$

$$\dot{s}_{zz} = (2Q + 2\eta D)\, w_z + p$$

Eq. 5.5 contd..

$$\dot{s}_{xy} = (Q + \eta D - \frac{\sigma_1 + \sigma_2}{2})(u_y + v_x) + \sigma_1 v_x$$

$$\dot{s}_{yx} = (Q + \eta D - \frac{\sigma_1 + \sigma_2}{2})(u_y + v_x) + \sigma_2 u_y$$

$$\dot{s}_{xz} = (Q + \eta D - \frac{\sigma_1}{2})(u_z + w_x) + \sigma_1 w_x$$

$$\dot{s}_{zx} = (Q + \eta D - \frac{\sigma_1}{2})(u_z + w_x)$$

$$\dot{s}_{yz} = (Q + \eta D - \frac{\sigma_2}{2})(v_z + w_y) + \sigma_2 w_y$$

$$\dot{s}_{zy} = (Q + \eta D - \frac{\sigma_2}{2})(v_z + w_y)$$

... (5.5)

where, $D \equiv \frac{\partial}{\partial t}$, and a subscript denotes partial differentiation. Also, the velocity v_i is replaced by its physical components u,v,w. It is to be noted that these velocity components are not independent but are connected through the incompressibility condition

$$u_x + v_y + w_z = 0 \qquad \text{... (5.6)}$$

The boundary conditions (5.4) at $z = \pm \frac{h}{2}$ suggest that

$$\left.\begin{aligned}
\dot{s}_{zz} &= (2Q + 2\eta D) w_z + p &&= 0 \\
\dot{s}_{zx} &= (Q + \eta D - \frac{\sigma_1}{2})(u_z + w_x) &&= 0 \\
\dot{s}_{zy} &= (Q + \eta D - \frac{\sigma_2}{2})(v_z + w_y) &&= 0
\end{aligned}\right\} \text{... (5.7)}$$

The first equation of (5.7) can be satisfied by imposing that $\dot{s}_{zz}$ is actually zero through the plate thickness.

The remaining two conditions in (5.7) are automatically satisfied by adopting the usual thin-plate assumptions that e_{31} and e_{32} are much smaller (and can be taken as zero) compared to in-plane strain components. Hence, the constitutive equations (5.5) take the following simplified form

$$
\begin{aligned}
\dot{s}_{xx} &= (2Q + 2\eta D - \sigma_1) u_x + p \\
\dot{s}_{yy} &= (2Q + 2\eta D - \sigma_2) v_y + p \\
\dot{s}_{zz} &= (2Q + 2\eta D) w_z + p \\
\dot{s}_{xy} &= (Q + \eta D - \frac{\sigma_1 + \sigma_2}{2}) (u_y + v_x) + \sigma_1 w_x \\
\dot{s}_{yx} &= (Q + \eta D - \frac{\sigma_1 + \sigma_2}{2}) (u_y + v_x) + \sigma_2 w_y \\
\dot{s}_{xz} &= \sigma_1 w_x \\
\dot{s}_{zx} &= 0 \\
\dot{s}_{yz} &= \sigma_2 w_y \\
\dot{s}_{zy} &= 0
\end{aligned}
\qquad \ldots(5.8)
$$

5.2 STABILITY CONDITION

As in the previous chapter, the criterion (1.43) is used for investigating the stability of the system. Substitution of the constitutive equations (5.8), along with the incompressibility condition (5.6) in (1.43) leads to the following condition for stability:

$$
\begin{aligned}
&\rho \int_0^a \int_0^b \int_{-\frac{h}{2}}^{\frac{h}{2}} [\, \dot{u}^2 + \dot{v}^2 + \dot{w}^2 \,]\ dx\ dy\ dz \\
&+ \int_0^a \int_0^b \int_{-\frac{h}{2}}^{\frac{h}{2}} [\ (2Q + 2\eta D - \sigma_1)\ u_x^2 \\
&+ (\, 2Q + 2\eta D - \sigma_2)\ v_y^2 + (\, 2Q + 2\eta D)\ w_z^2 \\
&+ (\, Q + \eta D - \frac{\sigma_1 + \sigma_2}{2})\ (u_y + v_x)^2 \\
&+ (\, \sigma_1\ v_x^2 + \sigma_2\ u_y^2\ ) + (\, \sigma_1\ w_x^2 + \sigma_2\ w_y^2\ )\ ]\ dx\ dy\ dz \\
&\qquad > 0 \qquad \ldots (5.9)
\end{aligned}
$$

An approximate velocity field is chosen which incorporates the usual engineering assumption that line elements perpendicular to the middle surface incipiently

remain straight and normal to the middle surface ; its familier form is

$$\left.\begin{aligned} u &= - z\, w_x \\ v &= - z\, w_y \\ w &= w\,(x,y,t) \end{aligned}\right\} \qquad \ldots \quad (5.10)$$

Substituting the velocity field (5.10) in the stability criterion (5.9), and subsequent integration through the thickness, results in

$$\frac{\rho\, h^3}{12} \int_0^a \int_0^b \left[\dot{w}_x^2 + \dot{w}_y^2 + \frac{12}{h^2}\, \dot{w}^2 \right] dx\, dy$$

$$+ \frac{2Q\, h^3}{12} \int_0^a \int_0^b \left[\left(1 + \frac{\eta\, D}{Q} - \frac{\sigma_1}{2Q} \right) w_{xx}^2 + \left(1 + \frac{\eta\, D}{Q} - \frac{\sigma_2}{2Q}\right) w_{yy}^2 \right.$$

$$\left. + \left(2 + \frac{2\eta\, D}{Q} - \frac{\sigma_1 + \sigma_2}{2Q}\right) w_{xy}^2 \right] dx\, dy$$

$$+ 2Q\, h \int_0^a \int_0^b \left[\frac{\sigma_1}{2Q}\, w_x^2 + \frac{\sigma_2}{2Q}\, w_y^2 \right] dx\, dy \;>\; 0$$

$$\ldots \quad (5.11)$$

Let the solution of w be taken in the following form

$$w = A_{mn} \sin \nu x \sin ky \exp\,(\bar{\omega}\, t) \qquad \ldots \quad (5.12)$$

where, $\nu = \frac{m\,\pi}{a}$, $k = \frac{n\,\pi}{b}$, $\bar{\omega} = i\,\omega$, m, n being positive integers, ω is the frequency and $i = \sqrt{-1}$. It is expected that this field will furnish a 'close' upper bound for the critical stress. Use of (5.12) in (5.11) results

in the following requirement for stability

$$\left[\frac{\rho\,\bar{\omega}^2}{2Q} \left(\nu^2 + k^2 + \frac{12}{h^2} \right) + \left(1 + \frac{\eta\,\bar{\omega}}{Q} \right) (\nu^2 + k^2)^2 \right]$$

$$+ \left[\left\{ \frac{12}{h^2} - (\nu^2 + k^2) \right\} \left(\frac{\sigma_1}{2Q}\, \nu^2 + \frac{\sigma_2}{2Q}\, k^2 \right) \right]$$

$$> 0 \qquad \ldots \quad (5.13)$$

which on simplification leads to

$$\frac{\dfrac{-\sigma_1}{2Q}}{\dfrac{P'}{1 - S'}} + \frac{\dfrac{-\sigma_2}{2Q}}{\dfrac{R'}{1 - S'}} < 1 \qquad \ldots \quad (5.14)$$

where,

$$P' = n^2\left(\frac{h}{b}\right)^2 \frac{\pi^2}{12} \left(1 + \frac{\eta\,\bar{\omega}}{Q}\right) \left(\frac{mb}{na} + \frac{na}{mb} \right)^2 + \frac{\rho\,\bar{\omega}^2 a^2}{2Q} \cdot \frac{1}{m^2 \pi^2} (1 + S')$$

$$R' = m^2\left(\frac{h}{a}\right)^2 \frac{\pi^2}{12} \left(1 + \frac{\eta\,\bar{\omega}}{Q}\right) \left(\frac{mb}{na} + \frac{na}{mb} \right)^2 + \frac{\rho\,\bar{\omega}^2 a^2}{2Q} \cdot \left(\frac{b}{a}\right)^2 \frac{1}{n^2 \pi^2} (1 + S')$$

and $$S' = \frac{\pi^2}{12} \left[m^2\left(\frac{h}{a}\right)^2 + n^2\left(\frac{h}{b}\right)^2 \right]$$

The inequality (5.13), after substituting $\bar{\omega} = i\,\omega$, can be rewritten in the following form

$$A\,\omega^2 - i\,B\,\omega - C < 0 \qquad \ldots \quad (5.15)$$

in which

$$A = \frac{\rho}{2Q} (\nu^2 + k^2 + \frac{12}{h^2}) \qquad \ldots \quad (5.15a)$$

$$B = \frac{\eta}{Q} (\nu^2 + k^2)^2 \qquad \ldots \quad (5.15b)$$

$$C = [\{ \frac{12}{h^2} - (\nu^2+k^2) \} (\frac{\sigma_1}{2Q} \nu^2 + \frac{\sigma_2}{2Q} k^2) + (\nu^2+k^2)^2] \qquad \ldots \quad (5.15c)$$

It is to be noted that A,B,C are real quantities while ω in general, is complex.

The inequality (5.14) further leads to

$$(\omega - \omega_1) \ (\omega - \omega_2) < 0 \qquad \ldots \quad (5.16)$$

where,

$$\omega_{1,2} = i \frac{B}{2A} \pm \frac{1}{2A} [- B^2 + 4AC]^{1/2} \qquad \ldots \quad (5.17)$$

Evidently, the first term in (5.17) represents the imaginary part of the frequency (and is normally positive when the motion is stable), while the second term in (5.17) represents the real part of the frequency if the expression within the square bracket is positive. Depending on the values of A,B,C, it is possible that the expression ($- B^2 + 4AC$) in (5.17) may be negative. In that case, the real part of the frequency is zero and the imaginary part of the frequency is non-zero. Alternatively, then,

$$\omega_{1,2} = i [\frac{B}{2A} \pm \frac{1}{2A} (B^2 - 4AC)^{1/2}] \qquad \ldots \quad (5.18)$$

Hence, for imaginary part of the frequency to be negative

$$\frac{B}{2A} < \frac{(B^2 - 4AC)^{1/2}}{2A}$$

which yields $4AC < 0$.

Since, $A > 0$, for imaginary part of the frequency to be negative,

$$C < 0 \qquad \ldots (5.19)$$

Using (5.15c), the preceding condition implies that

$$\left[\frac{12}{h^2} - \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right) \right] \left[\frac{\sigma_1}{2Q} \frac{m^2\pi^2}{a^2} + \frac{\sigma_2}{2Q} \frac{n^2\pi^2}{b^2} \right] + \left[\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right] < 0$$

which after simplification, can be rewritten as,

$$\frac{-\frac{\sigma_1}{2Q}}{n^2 \left(\frac{h}{b}\right)^2 E'} + \frac{-\frac{\sigma_2}{2Q}}{m^2 \left(\frac{h}{a}\right)^2 E'} > 1 \qquad \ldots (5.20)$$

where,

$$E' = \frac{\frac{\pi^2}{12} \left(\frac{mb}{na} + \frac{na}{mb} \right)^2}{\left[1 - \frac{\pi^2}{12} \left\{ m^2 \left(\frac{h}{a} \right)^2 + n^2 \left(\frac{h}{b} \right)^2 \right\} \right]}$$

Hence, for imaginary part of the frequency to be negative (i.e. when the motion is unstable), the condition (5.20) must be satisfied. Further, as evident from (5.18) and the

foregoing explanations whenever imaginary part of the frequency is negative, its real part is zero. Obviously, when the motion of the system passes from a stable state to an unstable one, due to increase in the stress ($\frac{\sigma_1}{2Q}$ and/or $\frac{\sigma_2}{2Q}$), the frequency passes through zero. Hence the critical value of stress would correspond to the situation when $\omega \equiv 0$. It is also unlikely that the imaginary part of the frequency will attain a negative value some where inbetween during gradual increase of stresses from zero to the value obtained from (5.20).

5.3 NUMERICAL RESULTS AND DISCUSSIONS

For numerical computations, the stresses, frequency and viscosity of the material are non-dimensionalised as follows:

$$\theta_1 = \frac{\sigma_1}{2Q} \;;\; \theta_2 = \frac{\sigma_2}{2Q} \;;\; \Omega = \sqrt{\left(\frac{\rho a^2}{2Q}\right)} \,.\, \omega \;;\quad \lambda_1 = \frac{\eta}{\sqrt{\left(\frac{\rho a^2 Q}{2}\right)}} \qquad \ldots (5.21)$$

The variables used in the numerical computations are

i) The plate dimensions characterized by $\frac{h}{b}$ (i.e. ratio of the thickness of the plate to its side in the y - direction) and aspect ratio $\frac{b}{a}$. The value of $\frac{h}{b}$ is taken as 0.01, while the ratio $\frac{b}{a}$ is varied.

ii) Material property is characterized by non-dimensional viscosity parameter λ_1 (as defined in Eq. 5.21). Three values of λ_1 (=0.01, 0.10 and 0.50) are taken to study its effect on the frequency parameter Ω.

iii) Two particular load-conditions are considered

a) $\theta_2 = 0$: the plate is loaded in the x-direction only

b) $\theta_1 \neq 0$, $\theta_2 \neq 0$: the plate is loaded biaxially.

When the plate is loaded in one direction, the critical value of the stress θ_1, as obtained from (5.20) is

$$\theta_1 = - n^2 \left(\frac{h}{b}\right)^2 \cdot \frac{\frac{\pi^2}{12}\left(\frac{mb}{na} + \frac{na}{mb}\right)^2}{\left[1 - \frac{\pi^2}{12}\left\{m^2\left(\frac{h}{a}\right)^2 + n^2 \left(\frac{h}{a}\right)^2\right\}\right]} \quad \ldots (5.22)$$

The above expression is minimum for n = 1 and various values of stress parameter θ_1 are plotted against the aspect ratio $\frac{a}{b}$ for different values of m (with n=1) in Fig. 5.1 . The lower envelop of these curves, as indicated by the thicker line, denotes the minimum value of the stress parameter θ_1 below which the plate is certainly stable.

When the plate is loaded biaxially critical value of the stresses θ_1 and θ_2 are determined from (5.14) which represents a system of straight lines in θ_1 and θ_2 with intercepts of $\frac{P'}{(1-S')}$ and $\frac{R'}{(1 - S')}$ on the abscissa and the ordinate respectively. The system of straight lines

in $\frac{\theta_1}{(\frac{h}{b})^2}$ and $\frac{\theta_2}{(\frac{h}{a})^2}$ are first plotted with different m and n (for a particular aspect ratio) and the critical envelop is determined. For various aspect ratio , the critical envelops are shown in Figs. 5.2 and 5.3 . The following observations can be made from these two figures:

1. Once the ratio $\frac{\theta_2}{\theta_1}$ (or $\frac{\sigma_2}{\sigma_1}$) and $\frac{b}{a}$ are fixed, it is easy to determine the stress parameters θ_2 and θ_1 which will cause instability.

2. It is obvious from Figs. 5.2 and 5.3 that the instability is impossible when parameters θ_1 and θ_2 are both tensile. But the possibility does remain when one of the stress parameters is tensile while the other one is compressive, as well as when both stress parameters are compressive.

3. Application of tensile stress (load) in one direction, increases the value of the critical stress.

Once the critical stress is determined for a particular value of $\frac{h}{a}$, $\frac{b}{a}$ and $\frac{\theta_2}{\theta_1}$, the stress parameter (and hence the load) is increased gradually from the value zero to its critical value and slightly beyond (say, in steps of $\frac{\theta_{cr}}{10}$). The frequency parameter Ω is determined for three values of viscosity parameter λ_1 (0.01, 0.10 and 0.50). These are tabulated in

Tables 5.1 to 5.3. The variation of real and imaginary parts of the frequency parameter with stress parameter, is shown schematically in Figs. 5.4 to 5.6. The eleventh row in Table 5.1 - 5.3 indicates the state of frequency parameter when the stress is just critical and the twelfth row shows that the imaginary parts of the frequency-parameters are negative when stress is more than the critical value. Also the real parts of the frequency-parameters are zero at these stress levels.

TABLE 5.1 : VARIATION OF FREQUENCY WITH STRESS IN UNIAXIALLY LOADED RECTANGULAR PLATE FOR

$\frac{h}{b} = 0.01; \quad \frac{\theta_2}{\theta_1} = 0 ; \quad m = 1, \; n = 1, \; \theta_{1cr} \; = \; \left(\frac{\sigma_1}{2Q}\right)_{cr} \; = -3.2904\text{x}10^{-4}$

$b/a = 1.0$

No.	Stress $\frac{\sigma_1}{2Q}$ $\text{x}10^{-4}$	Ω_1, Ω_2 $\text{x}10^{-2}$	Frequency with $\lambda_1 = 0.01$ $\text{x}10^{-5}$	Ω_3, Ω_4 $\text{x}10^{-2}$	Frequency with $\lambda_1 = 0.10$ $\text{x}10^{-4}$	Ω_5, Ω_6 $\text{x}10^{-2}$	Frequency with $\lambda_1 = 0.50$ $\text{x}10^{-4}$
1	0.	± 5.698	1.623	± 5.698	1.623	± 5.697	8.116
2	-0.329	± 5.405	1.623	± 5.405	1.623	± 5.405	8.116
3	-0.658	± 5.096	1.623	± 5.096	1.623	± 5.096	8.116
4	-0.987	± 4.767	1.623	± 4.762	1.623	± 4.766	8.116
5	-1.316	± 4.413	1.623	± 4.413	1.623	± 4.413	8.116
6	-1.645	± 4.029	1.623	± 4.029	1.623	± 4.028	8.116
7	-1.974	± 3.604	1.623	± 3.604	1.623	± 3.603	8.116
8	-2.303	± 3.121	1.623	± 3.121	1.623	± 3.120	8.116
9	-2.632	± 2.548	1.623	± 2.548	1.623	± 2.547	8.116
10	-2.961	± 1.802	1.623	± 1.802	1.623	± 1.800	8.116
11	-3.290	0. 0.	0. 3.246	0. 0.	0. 3.246	0. 0.	0. 16.23
12	-3.619	0. 0.	-1800 1803	0. 0.	-178.6 181.8	0. 0.	-172.2 188.5

TABLE 5.2 : VARIATION OF FREQUENCY WITH STRESS IN UNIAXIALLY LOADED RECTANGULAR PLATE FOR

$\frac{h}{b} = 0.01, \quad \frac{a}{b} = 1.5 ; \quad \frac{\theta_2}{\theta_1} = 0 ; \; m = 2, \; n=1 ; \quad \theta_{1cr} = \left(\frac{\sigma_1}{2Q}\right)_{cr} = -3.571 \times 10^{-4}$

No.	Stress $\frac{\sigma_1}{2Q}$ $\times 10^{-4}$	Frequency Ω_1, Ω_2 with $\lambda_1=0.01$ $\times 10^{-1}$	$\times 10^{-5}$	Frequency Ω_3, Ω_4 with $\lambda_1=0.10$ $\times 10^{-1}$	$\times 10^{-4}$	Frequency Ω_5, Ω_6 with $\lambda_1=0.50$ $\times 10^{-1}$	$\times 10^{-3}$
1	0.	$\pm$ 1.187	7.045	$\pm$ 1.187	7.045	$\pm$ 1.186	3.522
2	-0.357	$\pm$ 1.126	7.045	$\pm$ 1.126	7.045	$\pm$ 1.126	3.522
3	-0.714	$\pm$ 1.062	7.045	$\pm$ 1.062	7.045	$\pm$ 1.061	3.522
4	-1.071	$\pm$ 0.993	7.045	$\pm$ 0.993	7.045	$\pm$ 0.993	3.522
5	-1.428	$\pm$ 0.919	7.045	$\pm$ 0.919	7.045	$\pm$ 0.919	3.522
6	-1.785	$\pm$ 0.839	7.045	$\pm$ 0.839	7.045	$\pm$ 0.839	3.522
7	-2.142	$\pm$ 0.751	7.045	$\pm$ 0.751	7.045	$\pm$ 0.745	3.522
8	-2.499	$\pm$ 0.650	7.045	$\pm$ 0.650	7.045	$\pm$ 0.649	3,522
9	-2.856	$\pm$ 0.531	7.045	$\pm$ 0.531	7.045	$\pm$ 0.530	3.522
10	-3.214	$\pm$ 0.375	7.045	$\pm$ 0.375	7.045	$\pm$ 0.734	3.522
11	-3.571	0. 0.	0. 14.09	0. 0.	0. 14.09	0. 0.	0. 7.045
12	-3.928	0. 0.	-374 .7 3761	0. 0.	-368.4 382.5	0. 0.	-34.18 41.22

TABLE 5.3 : VARIATION OF FREQUENCY WITH STRESS IN BIAXIALLY LOADED RECTANGULAR PLATE

FOR $\frac{h}{b} = 0.01$; $\frac{b}{a} = 1.0$; $\frac{\theta_2}{\theta_1} = 1.0$; $m = 1$, $n = 1$, $Q_{1cr} = \left(\frac{\sigma_1}{2Q}\right)_{cr} = -1.6452 \times 10^{-4}$

No.	Stress $\frac{\sigma_1}{2Q}$ $\times 10^{-4}$	Frequency Ω_1, Ω_2 with $\lambda_1 = 0.01$ $\times 10^{-2}$	$\times 10^{-5}$	Frequency Ω_3, Ω_4 with $\lambda_1 = 0.10$ $\times 10^{-2}$	$\times 10^{-4}$	Frequency Ω_5, Ω_6 with $\lambda_1 = 0.50$ $\times 10^{-2}$	$\times 10^{-2}$
1	0.	$\pm$ 5.698	1.623	$\pm$ 5.698	1.623	$\pm$ 5.697	8.116
2	-0.1645	$\pm$ 5.405	1.623	$\pm$ 5.405	1.623	$\pm$ 5.405	8.116
3	-0.3290	$\pm$ 5.096	1.623	$\pm$ 5.096	1.623	$\pm$ 5.096	8.116
4	-0.4936	$\pm$ 4.767	1.623	$\pm$ 4.767	1.623	$\pm$ 4.766	8.116
5	-0.6581	$\pm$ 4.413	1.623	$\pm$ 4.413	1.623	$\pm$ 4.413	8.116
6	-0.8226	$\pm$ 4.029	1.623	$\pm$ 4.029	1.623	$\pm$ 4.028	8.116
7	-0.9871	$\pm$ 3.604	1.623	$\pm$ 3.604	1.623	$\pm$ 3.603	8.116
8	-1.1516	$\pm$ 3.121	1.623	$\pm$ 3.121	1.623	$\pm$ 3.120	8.116
9	-1.3162	$\pm$ 2.518	1.623	$\pm$ 2.548	1.623	$\pm$ 2.547	8.116
10	-1.4807	$\pm$ 1.802	1.623	$\pm$ 1.802	1.623	$\pm$ 1.800	8.116
11	-1.6452	0. 0.	0. 3.246	0. 0.	0. 3.246	0. 0.	0. 16.23
12	-1.8097	0. 0.	-1800 1.803	0. 0.	-178.6 1.818	0. 0.	-17.22 1.885

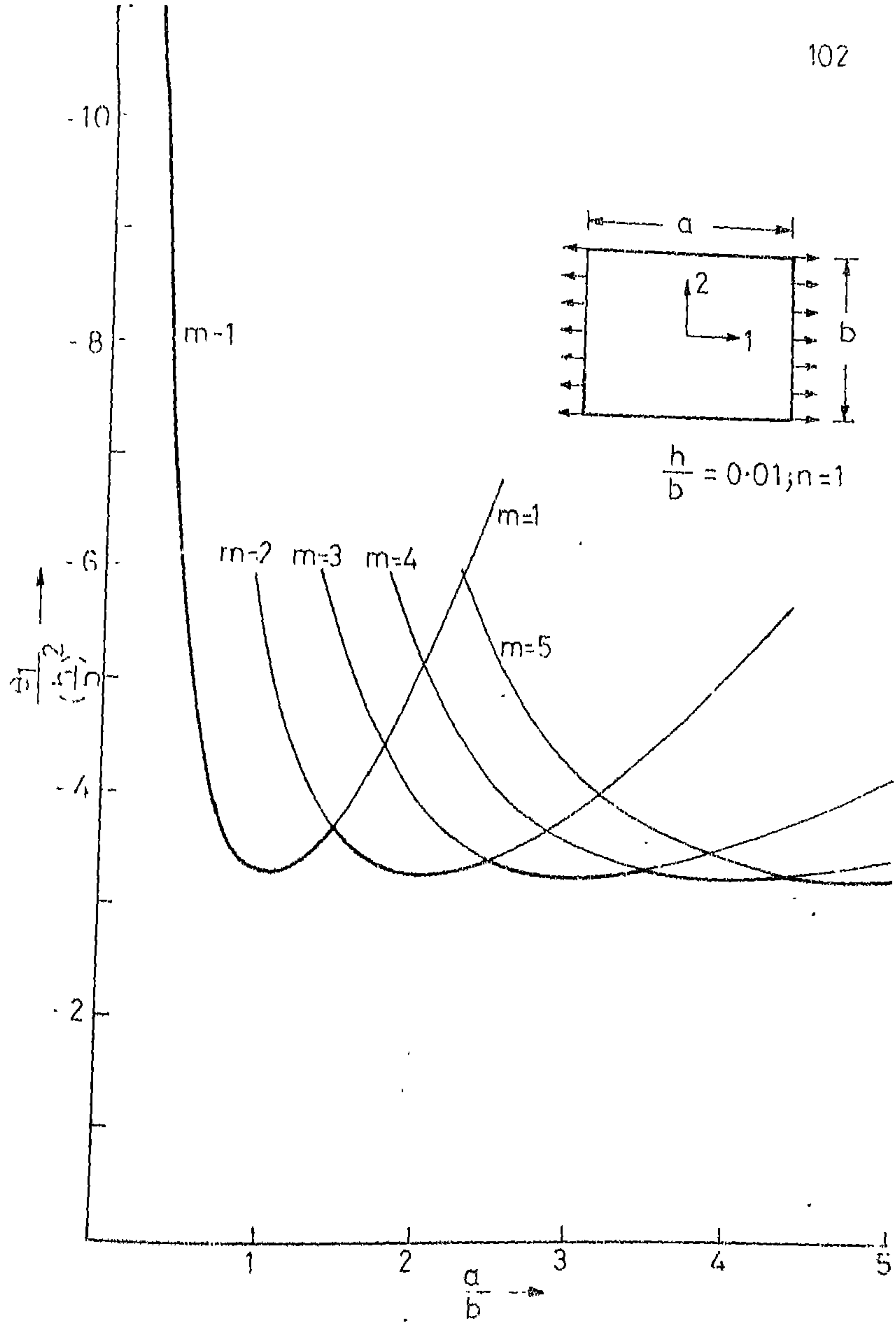


Fig.5·1 Critical stress vs aspect ratio; uniaxially loaded rectangular plate

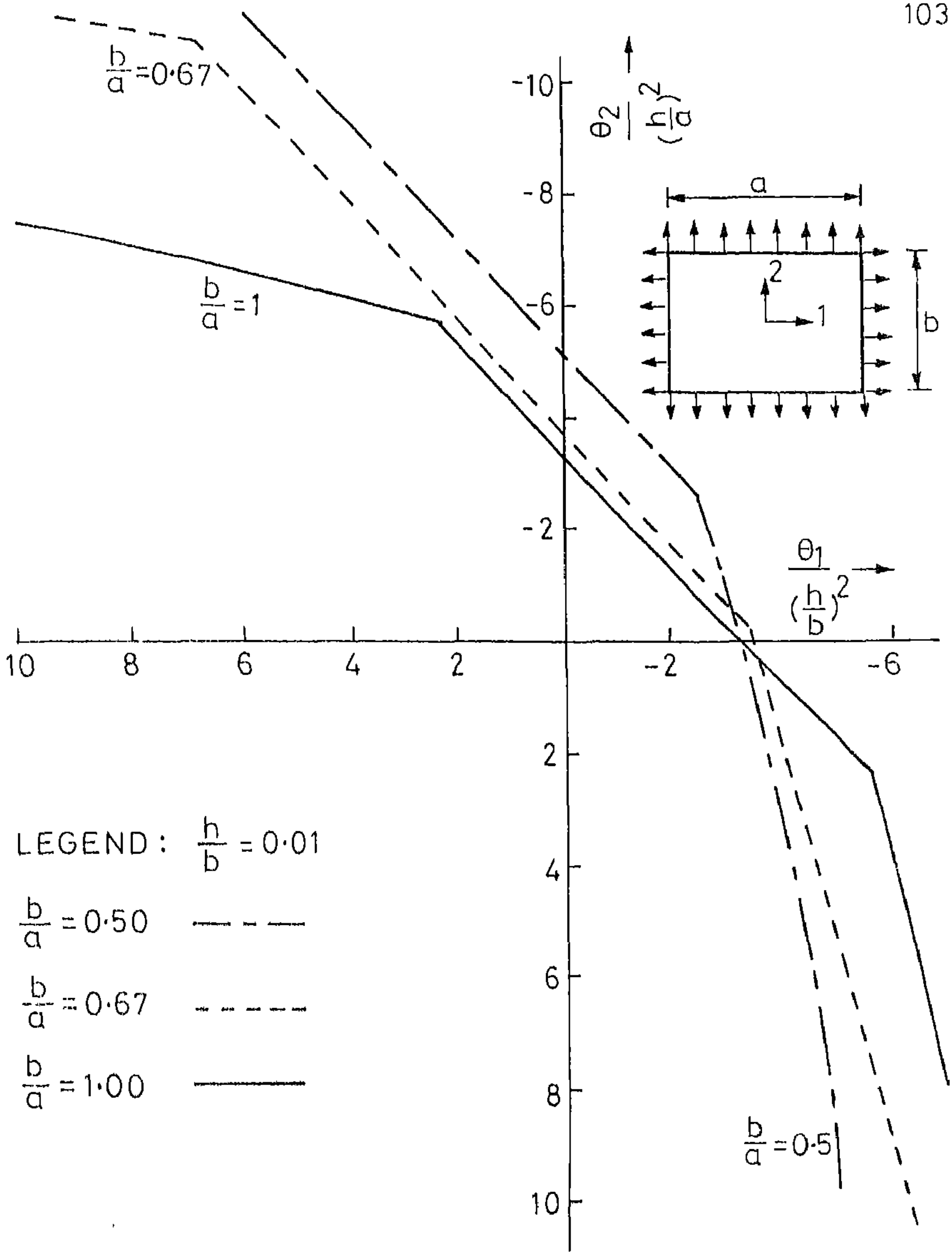


Fig.5·2 Plot of the critical envelope of biaxially loaded rectangular plate

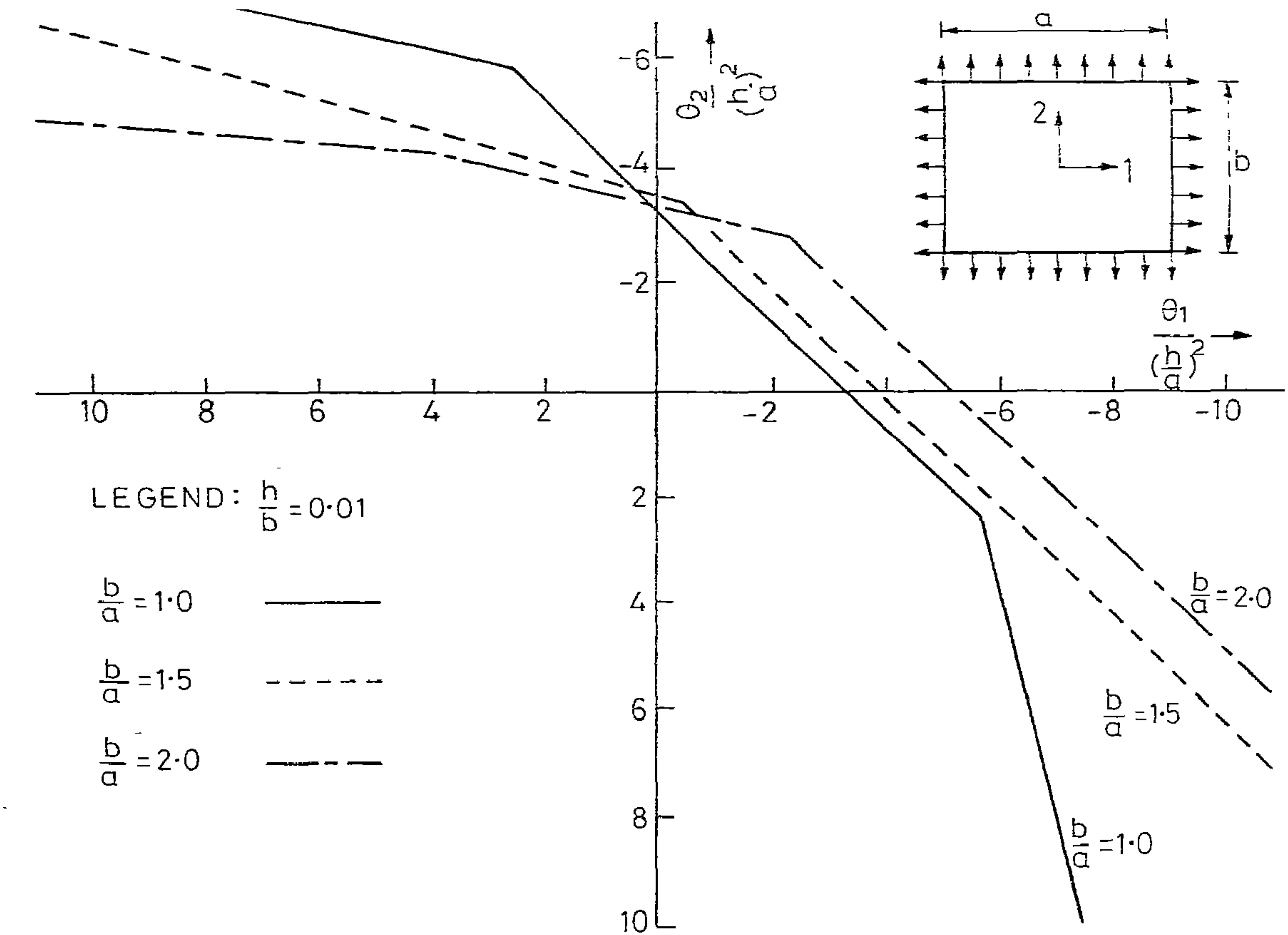


Fig.5·3 Plot of the critical envelope of biaxially loaded rectangular plate

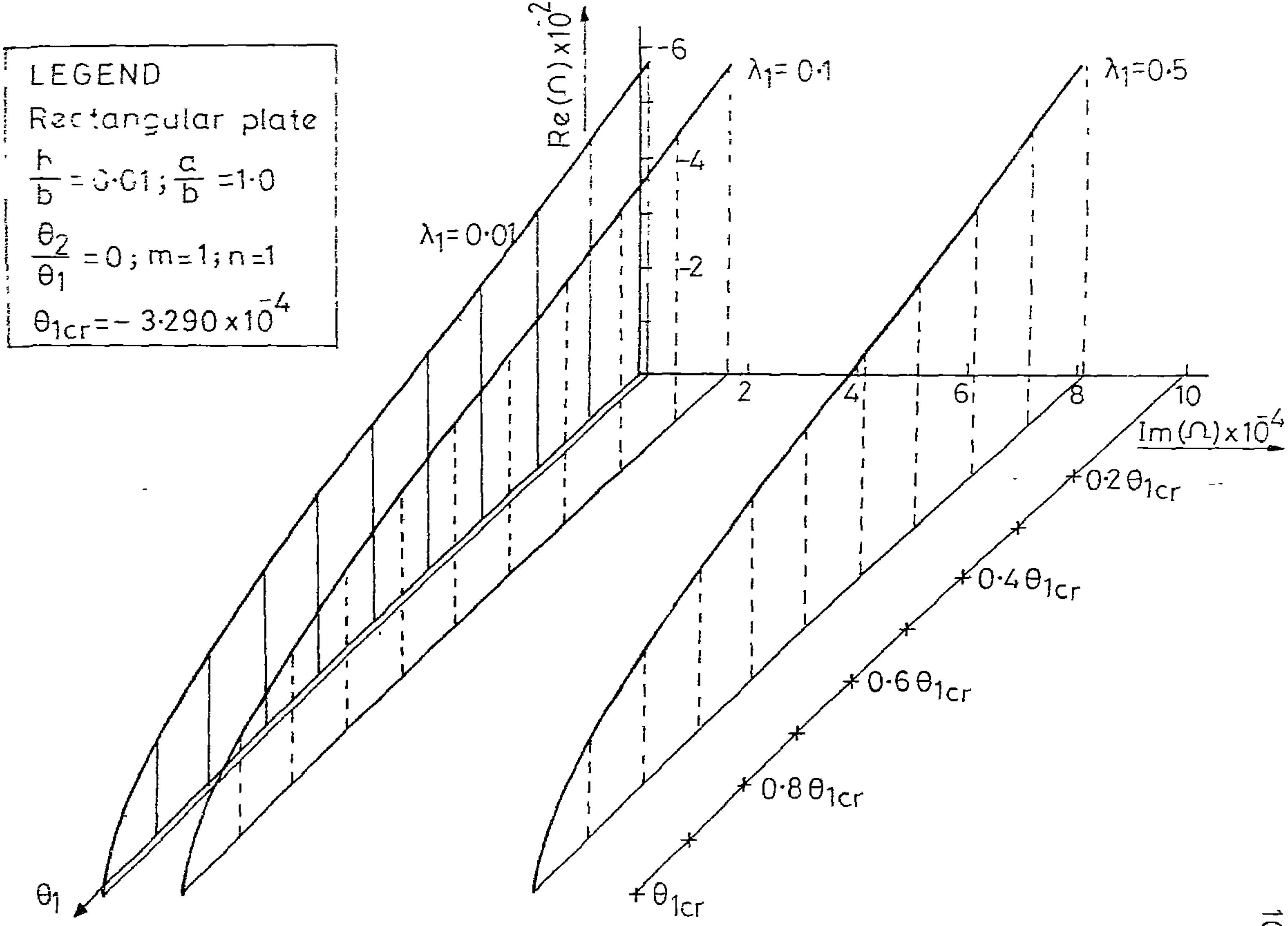


Fig.5·4 Variation of frequency with stress

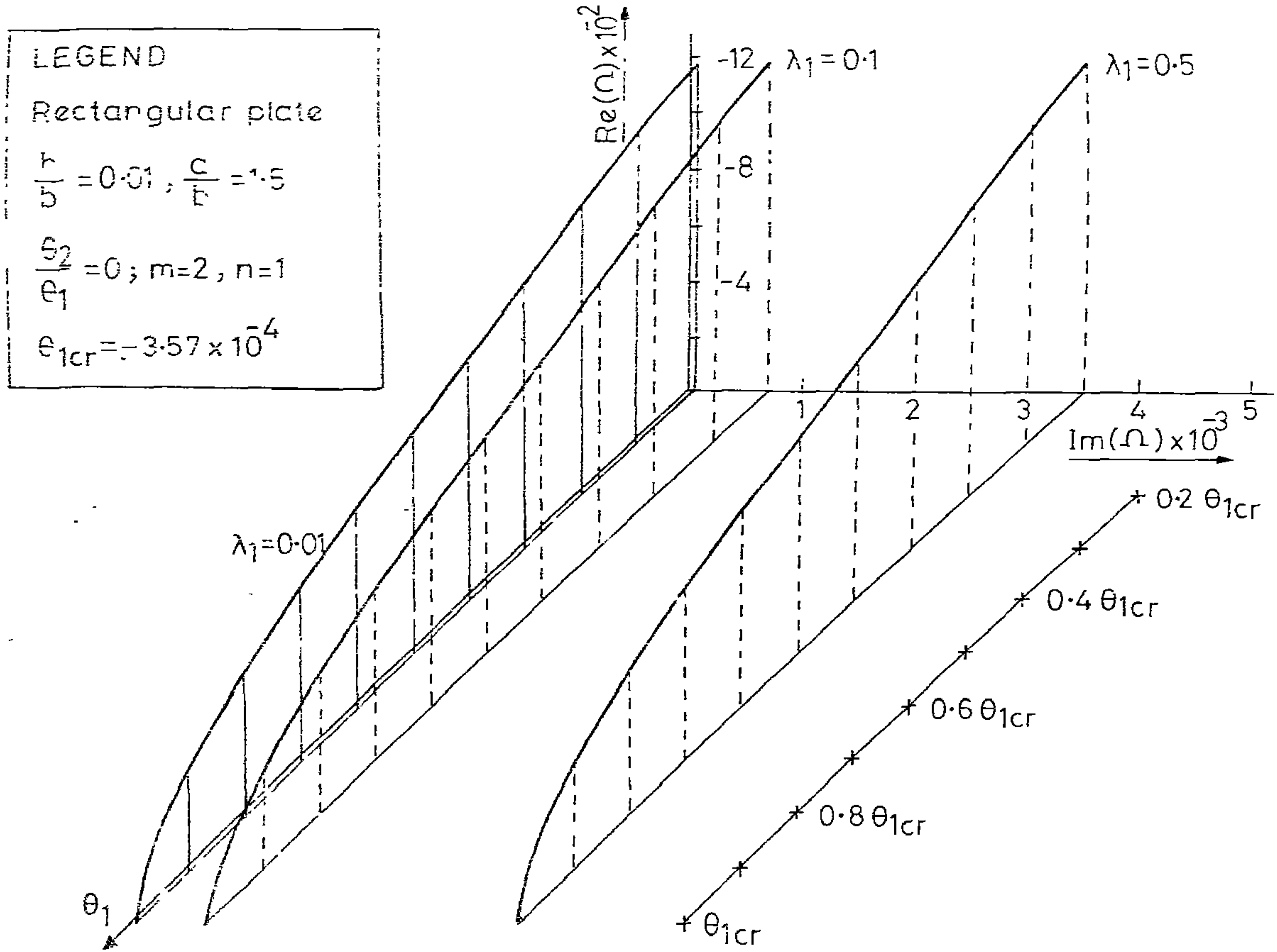


Fig.5.5 Variation of frequency with stress

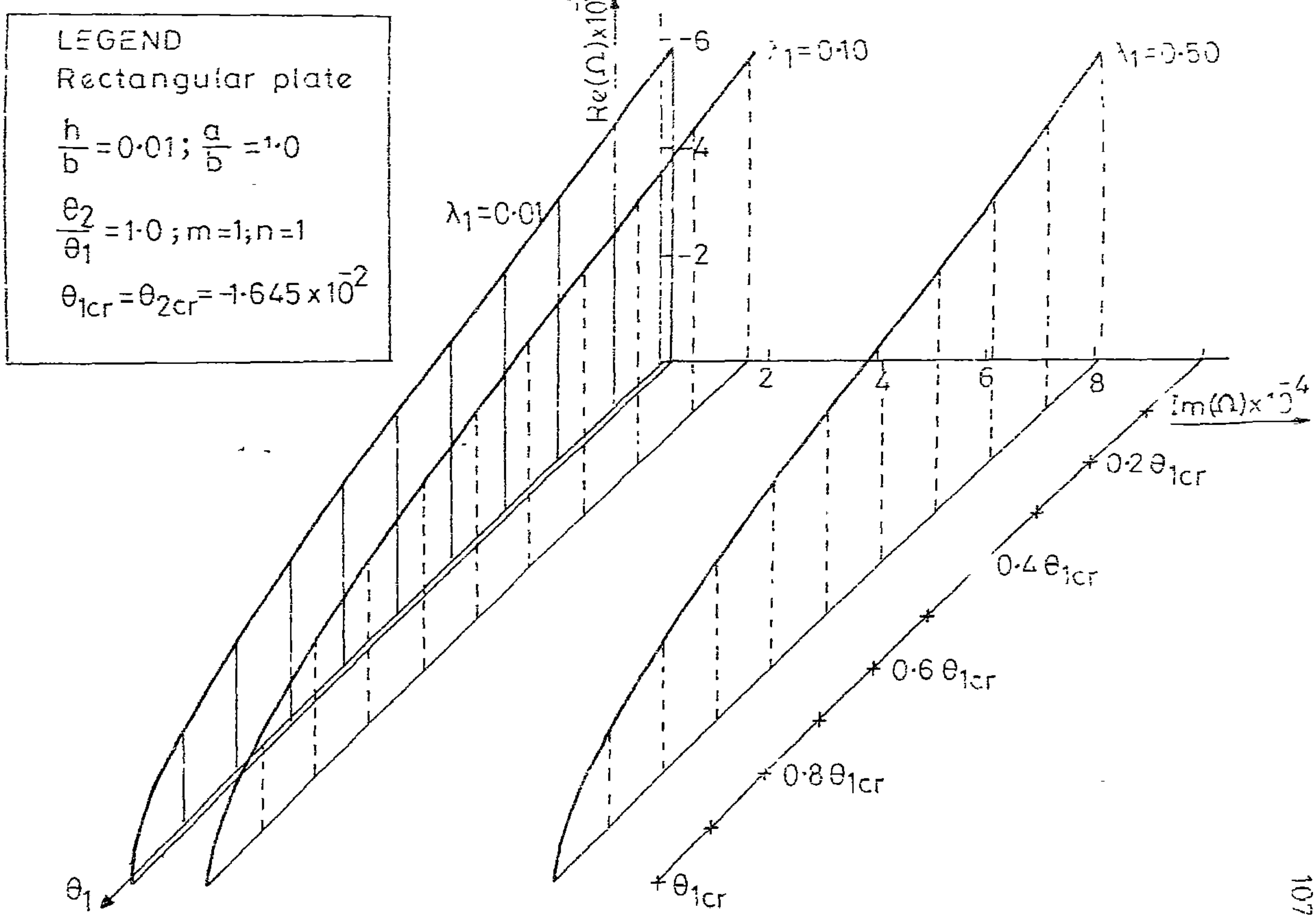


Fig.5·6 Variation of frequency with stress

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CONCLUSIONS

The present investigation has been concerned with some stability problems in a finitely deformed incompressible linear viscoelastic solid. Conditions for instability have been obtained in all the problems studied.

Exhaustive formulation has been presented to investigate instability in a rectangular solid in plane strain. In this case, the numerical solution of the instability condition shows that the only possible mode of instability is the antisymmetric one for all possible material dimensions. The analytical expressions derived for instability in the symmetric mode yield higher value of the critical stress than those obtained from the corresponding expressions in the antisymmetric mode and are, therefore, of only academic interest. The results obtained for the limiting case of the surface instability in a viscous medium have been found identical to those of Biot (1965).

The condition for axisymmetric instability has been derived for an incompressible viscoelastic solid cylinder of finite dimensions under axial loading. As resulting condition is quite involved, it has not been possible to

rule out instability under tensile loading. However, the numerical solution of the condition yields only the compressive values of critical stress for different cylinder dimensions.

For a long thin viscoelastic cylindrical shell the value of the critical shear stress (and hence the torque) has been obtained at which the shell becomes unstable. These have been plotted for different radius-thickness ratios.

In the case of thin rectangular viscoelastic plates, the conditions for instability have been examined both under axial and biaxial loading. The instability due to unidirectional in-plane tensile loading has been ruled out. Critical stresses have been obtained for the axial and the biaxial loadings for various aspect ratios. In case of biaxial loading, the interaction curves have been plotted to illustrate the region of stability/instability.

It has been observed in all the problems that, as the stress/load gradually approaches its critical value and goes slightly beyond, the frequency of the system passes through the value zero. In other words, when the stress/load is below the critical value, the real part of the frequency is non-zero while its imaginary part is positive. The stage at which both the real and imaginary parts of the frequency are zero is termed as the critical one because it marks the

transition from a stable state to an unstable state. When the stress/load attains a value higher than the critical one, the real part of the frequency remains zero while its imaginary part becomes negative. This indicates divergent type of motion and instability of the system.

It is obvious that the loss of stability is of static type. This gives an impression that the viscosity parameter has no effect. But, this is not so. The critical stress is in terms of a material parameter Q which actually is a function of the current stress. In fact, the critical stress must be expressed in terms of the material parameter Q_o in the underformed configuration. The relationship between Q and Q_o depends upon the manner in which the body has been finitely deformed (see, e.g. Biot, 1965) and it, naturally, is dependent on viscosity.

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